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I.—ELIMINATION BY INDETERMINATE MULTIPLIERS.

As we have made frequent use of the method of elimination by means of indeterminate multipliers, we shall give the theory of the process.

The solution of an extensive class of problems may be reduced to the determination of the maximum or minimum of a function U of n variables, $x_1, x_2, \ldots\ldots x_n$; these being in general not independent but subject to certain equations of condition,

$$L_1 = 0,\ L_2 = 0, \ldots\ldots L_r = 0.$$

The method simplest in theory, is to determine, by means of the r equations of condition, r of the quantities x, as for instance, $x_1, x_2, \ldots\ldots x_r$, in terms of the remaining x's; substituting these values, U will now be a function of the $n - r$ independent variables $x_{n-r} \ldots\ldots x_n$; and making the coefficient of each differential in dU equal to zero, we get $n - r$ equations, which, with the r equations of condition, determine $x_1, x_2, \ldots\ldots x_n$.

But in practice this method is rarely possible, and still more rarely convenient. Instead of this, we may consider each of the n quantities x to vary in U, their variations being no longer independent, but subject to the condition that in their varied state they must satisfy the r equations $L = 0$, or that $L_1 + dL_1 = 0$, &c., and as $L_1, L_2 \ldots\ldots$ are $= 0$, the variations must be such as to satisfy the r equations

$$dL_1 = 0,\quad dL_2 = 0, \ldots\ldots dL_r = 0.$$

From these equations we may determine r of the variations in terms of the $(n - r)$ remaining; and substituting these in dU, we shall have, as before, the coefficients of the $(n - r)$ variations to equate to zero.

The simplest method of effecting this is the following. The quantities dL_1, dL_2, ... as well as dU, are linear in dx_1, dx_2 ..... being of the form

$$\frac{dL}{dx_1}dx_1 + \frac{dL}{dx_2}dx_2 \ldots\ldots \frac{dL}{dx_n}dx_n.$$

Multiplying them by λ_1, λ_2, ... λ_r respectively, and adding them to dU, we have

$$dU + \lambda_1 dL_1 + \lambda_2 dL_2 \ldots\ldots + \lambda_r dL_r = 0,$$

an equation which will be of the form

$$M_1 dx_1 + M_2 dx_2 \ldots\ldots + M_n dx_n = 0,$$

in which each quantity $M = \dfrac{dU}{dx} + \lambda_1 \dfrac{dL_1}{dx} + \lambda_2 \dfrac{dL_2}{dx}$.

If we determine the r quantities λ by the conditions that they must make the coefficients of the variations dx_1, dx_2, ... dx_r equal to zero, that is to say, if we determine them from the equations $M_1 = 0$ ..... $M_r = 0$, we shall have effected our purpose. For by this means the variations dx_1 ..... dx_r are eliminated, and there remains

$$M_{r+1}dx_{r+1} \ldots\ldots + M_n dx_n = 0;$$

and as these $n - r$ variations are independent, their coefficients must be equal to zero.

Hence, if we determine λ_1, λ_2, ... λ_r from the equations

$$M_1 = 0 \ldots\ldots M_r = 0,$$

we shall also have

$$M_{r+1} = 0 \ldots\ldots M_n = 0.$$

Thus the r quantities λ and the n quantities x satisfy the $n + r$ equations

$$M_1 = 0,\ M_2 = 0, \ldots\ldots M_n = 0,\ L_1 = 0, \ldots\ldots L_r = 0.$$

And as it is indifferent which of the variations we suppose to have been eliminated, and which set of the equations $M = 0$ to have been used for the determination of the λ's, the most general way of stating the result is, that we have the $n + r$ equations

$$L_1 = 0,\ L_2 = 0, \ldots\ldots L_r = 0,$$

$$\left.\begin{aligned} \frac{dU}{dx_1} + \lambda_1 \frac{dL_1}{dx_1} + \lambda_2 \frac{dL_2}{dx_1} \ldots\ldots \lambda_r \frac{dL_r}{dx_1} &= 0, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot & \\ \frac{dU}{dx_n} + \lambda_1 \frac{dL_1}{dx_n} + \lambda_2 \frac{dL_2}{dx_n} \ldots\ldots \lambda_r \frac{dL_r}{dx_n} &= 0, \end{aligned}\right\} \ldots(1),$$

to determine the $n + r$ quantities x_1, x_2 ..... x_n, λ_1 ..... λ_r.

If U be homogeneous in x_1, x_2 ....., and if L_1, L_2 ..... consist of terms of not more than two different dimensions, there exists a simple relation between the quantities λ, which is very useful in many problems.

Suppose U to be homogeneous of u dimensions, L_1, L_2, &c. to consist of terms of two different dimensions, so that the equations $L_1 = 0$, $L_2 = 0$, &c. may be put under the form

$$M_a + A = 0, \quad N_b + B = 0, \text{ &c.}$$

where M_a is homogeneous of a dimensions, and A a constant.

Then multiplying the equations (1) by x_1, x_2, ... and adding, we have, by the property of homogeneous functions,

$$0 = uU + a\lambda_1 M_a + b\lambda_2 N_b + \text{&c.}$$
$$\text{or } uU' = aA\lambda_1 + bB\lambda_2 + cC\lambda_3 + \text{&c.}$$

We shall now proceed to give a few examples of the application of this method.

If we take the tangent plane at any point in a surface for the plane of xy, and the normal for the axis of z; and if r, s, t, according to the usual notation, represent the values of $\frac{d^2z}{dx^2}$, $\frac{d^2z}{dx\,dy}$, $\frac{d^2z}{dy^2}$, at the point in question, α, β the cosines of the angles that a plane passing through the axis of z makes with the planes of xz, yz, and if ρ be the radius of curvature of the section of the surface made by this plane, it is known that

$$\frac{1}{\rho} = r\alpha^2 + 2s\alpha\beta + t\beta^2 \text{.....(1)}$$

with the relation $\qquad \alpha^2 + \beta^2 = 1$.....(2).

If we wish to find the direction of the section of greatest or least curvature, we may substitute $\sqrt{1-\alpha^2}$ for β; we have then

$$\frac{1}{\rho} = (r - t)\alpha^2 + zs\alpha\sqrt{1-\alpha^2} + t,$$

from which we may determine the maximum and minimum values of ρ with the corresponding directions.

But without eliminating β, we may consider it as a function of α given by equation (2), and make α and β vary in (1), subjecting them to the condition of satisfying (2) in their varied state.

Hence

$$0 = (r\alpha + s\beta)\,d\alpha + (s\alpha + t\beta)\,d\beta \text{.....(3)},$$
$$0 = \alpha\,d\alpha + \beta\,d\beta \text{.....(4)}.$$

If we eliminate $d\beta$, the coefficient of $d\alpha$ equated to zero will give a relation between α and β; to do this, multiply (4) by λ, and add to (3)

$$(r\alpha + s\beta + \lambda\alpha)\,d\alpha + (s\alpha + t\beta + \lambda\beta)\,d\beta = 0.$$

If, then, we determine λ from the condition

$$s\alpha + t\beta + \lambda\beta = 0 \text{.....(5)},$$

we must also have

$$r\alpha + s\beta + \lambda\alpha = 0 \text{.....(6)}.$$

These equations give us at once

$$s\frac{\alpha}{\beta} + t = -\lambda = r + s\frac{\beta}{\alpha},$$

$$\text{or } \frac{\alpha^2}{\beta^2} - \frac{r-t}{s}\frac{\alpha}{\beta} - 1 = 0 \ldots\ldots\ldots(7),$$

which shows us, that the two directions of maximum and minimum curvature are at right angles to each other.

Multiplying by α and β, and adding,

$$r\alpha^2 + 2s\alpha\beta + t\beta^2 + \lambda(\alpha^2 + \beta^2) = 0,$$

$$\text{or } \frac{1}{\rho} + \lambda = 0 \ldots\ldots\ldots\ldots \ldots(8).$$

Substituting this value, (5) and (6) may be put in the form

$$\left(\frac{1}{\rho} - t\right)\beta = s\alpha,$$

$$\left(\frac{1}{\rho} - r\right)\alpha = s\beta,$$

and multiplying together,

$$\left(\frac{1}{\rho} - r\right)\left(\frac{1}{\rho} - t\right) - s^2 = 0.$$

The relation $\frac{1}{\rho} + \lambda = 0$, corresponds to the relation $u\mathrm{U} = a\alpha\lambda_1$ + &c., which we have found in the general case.

It may be observed in this case, that if we merely wish to determine the relation between α and β, we may eliminate $d\beta$ at once and obtain the relation; and if we have three equations and three arbitrary variations, we should get the relation by eliminating two of them by the method of cross multiplication. An example of this will be found in the investigation in p. 7 of our last Number.

For a second example, let us investigate the properties of conjugate diameters in surfaces of the second order.

Let

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{z^2}{c'^2} = 1 \ldots\ldots\ldots\ldots\ldots(1)$$

be the equation of the surface referred to any three conjugate diameters, whose lengths are a', b', c'.

If l, m, n be the cosines of the angles which the axes make with each other, r any radius,

$$r^2 = x^2 + y^2 + z^2 + 2lyz + 2mxz + 2nyz \ldots\ldots\ldots(2).$$

When r coincides with one of the principal axes, it is a maximum or minimum, and we have the condition $dr = 0$.

If we determine z from (1), and substitute in (2), r is a function of x and y only, and hence we should have

$$\frac{d(r)}{dx} = 0, \quad \frac{d(r)}{dy} = 0;$$

which, with (1) and (2), will serve to determine x, y, z, r.

This would be very difficult; but we may, as before, differentiate (1) and (2), considering x, y, z to vary.

We thus get

$$0 = (x + mz + ny)\,dx + (y + lz + nx)\,dy + (z + ly + mx)\,dz,$$

$$0 = \frac{x}{a'^2}\,dx + \frac{y}{b'^2}\,dy + \frac{z}{c'^2}\,dz.$$

If we multiply the second by λ and add to the first, and determine λ from the condition

$$\lambda \frac{z}{c'^2} + z + ly + mx = 0,$$

we have an equation in which dz has been eliminated; and as dx, dy are independent, their coefficients must also be equal to zero, or

$$\lambda \frac{y}{b'^2} + y + lz + nx = 0,$$

$$\lambda \frac{x}{a'^2} + n + mz + ny = 0.$$

If we multiply these by x, y, z respectively and add, we obtain, in consequence of (1) and (2),

$$\lambda + r^2 = 0.$$

Substituting this value of λ in the equations, they become

$$\left(1 - \frac{r^2}{a^2}\right) x + ny + mz = 0,$$

$$nx + \left(1 - \frac{r^2}{b^2}\right) y + lz = 0,$$

$$mx + ly + \left(1 - \frac{r^2}{c^2}\right) z = 0.$$

Eliminating x, y, z between these by the method of cross multiplication, the resulting equation is

$$\left(1 - \frac{r^2}{a^2}\right)\left(1 - \frac{r^2}{b^2}\right)\left(1 - \frac{r^2}{c^2}\right) - l^2\left(1 - \frac{r^2}{a^2}\right) - m^2\left(1 - \frac{r^2}{b^2}\right) - n^2\left(1 - \frac{r^2}{c^2}\right) + 2lmn = 0,$$

a cubic equation in r^2, the roots of which are the squares of the principal axes; and by a discussion of which the properties of the conjugate diameters may be deduced.—Vide *Hymers'* Analytical Geometry.

As another example, find the values of x, y, z which may make the function

$$u = (x + 1)(y + 1)(z + 1) \text{ a maximum,}$$

x, y, z being connected by the equation

$$a^x \cdot b^y \cdot c^z = \mathrm{A}.$$

Taking the logarithmic differentials of both equations,

$$0 = \frac{dx}{x+1} + \frac{dy}{y+1} + \frac{dz}{z+1},$$

$$0 = dx \log a + dy \log b + dz \log c.$$

Multiply the second by $-\lambda$, add and equate to 0 the coefficients of each differential, then

$$\lambda \log a = \frac{1}{x+1}, \quad \lambda \log b = \frac{1}{y+1}, \quad \lambda \log c = \frac{1}{z+1}.$$

Multiply by $x+1$, $y+1$, $z+1$, and add; then

$$\lambda (x \log a + y \log b + z \log c + \log a + \log b + \log c) = 3,$$

$$\text{or } \lambda \log (\mathrm{A}abc) = 3.$$

Therefore substituting

$$x+1 = \frac{\log (\mathrm{A}abc)}{3 \log a}, \quad (y+1) = \frac{\log (\mathrm{A}abc)}{3 \log b}, \quad z+1 = \frac{\log (\mathrm{A}abc)}{3 \log c};$$

therefore

$$u = (x+1)(y+1)(z+1) = \frac{\log (\mathrm{A}abc)^3}{\log (abc)^3}.$$

M. S.

II.—ON THE SOLUTION OF LINEAR EQUATIONS OF FINITE AND MIXED DIFFERENCES.

In the preceding Number the principle of the separation of the symbols of operation from those of quantity, was applied to the solution of Differential Equations. We shall here apply the same principle to equations of Finite and Mixed Differences; but, as the method is not very different from that previously delivered, there will be no necessity for dwelling long on it.

The general form of the linear equation of Finite Differences with constant coefficients, is

$$u_{x+n} + \mathrm{A}u_{x+n-1} + \&\text{c.} + \mathrm{R}u_{x+1} + \mathrm{S}u_x = \mathrm{X},$$

where A, B, C, &c. are constants, and X a function of x. By a known relation we can express the quantities u_{x+n}, u_{x+n-1}, &c. in

terms of u_x and its successive differences, so that the equation may be transformed into one which may be put under the form

$$f(\Delta)\, u_x = \mathrm{X};$$

which we might proceed to solve by means of the separation of the symbols. But it will be more convenient to proceed in the following manner. Since we have, generally,

$$u_{x+n} = (1 + \Delta)^n u_x,$$

we may consider $1 + \Delta$ as a separate symbol, subject to the same laws as d and Δ; the effect of which, when applied on u_x, is to convert u_x into u_{x+1}. If we represent, for the sake of shortness, $1 + \Delta$ by D, the equation will take the form

$$(\mathrm{D}^n + \mathrm{AD}^{n-1} + \&\mathrm{c.} + \mathrm{RD} + \mathrm{S})u_x = \mathrm{X},$$

or, as we may write it for convenience,

$$\mathrm{F}(\mathrm{D})\, u_x = \mathrm{X}.$$

Then, converting $\mathrm{F}(\mathrm{D})$ into factors of the form $\mathrm{D} - a$, a being one of the roots of the equation $\mathrm{F}(z) = 0$, we can perform the inverse operation for each factor separately, and so arrive at the value of u_x. To do this more readily, we shall avail ourselves of a theorem similar to that given in page 25. This theorem is, that

$$(\mathrm{D} - a)^n \mathrm{X} = a^{n+x} \Delta^n (\mathrm{X} a^{-x}).$$

For if we expand the first side, it may be put under the form

$$a^n (\mathrm{D}^n a^{-n} - n\mathrm{D}^{n-1} a^{-(n-1)} + \frac{n.n-1}{1.2} \mathrm{D}^{n-2} a^{-(n-2)} - \&\mathrm{c.})\, \mathrm{X}.$$

$$\text{Now } \mathrm{D}^p a^{-x} = a^{-(x+p)}, \text{ and } a^{-p} = a^x \mathrm{D}^p a^{-x};$$

so that the expression may be put under the form

$$a^{n+x} (\mathrm{D}^n \mathrm{D}'^n - n\mathrm{D}^{n-1} \mathrm{D}'^{n-1} + \&\mathrm{c.})\, \mathrm{X} a^{-x},$$

where the accented letters refer to a^{-x} and the unaccented to X. This expression is equivalent to

$$a^{n+x} (\mathrm{DD}' - 1)^n \mathrm{X} a^{-x}$$

$$= a^{n+x} (\Delta + \Delta' + \Delta\Delta')^n \mathrm{X} a^{-x} = a^{n+x} \Delta^n (\mathrm{X} a^{-x}),$$

by a theorem analogous to that of Leibnitz.

To apply this to the general equation $\mathrm{F}(\mathrm{D})\, u_x = \mathrm{X}$, suppose $\mathrm{F}(\mathrm{D})$ to be resolved into binomial factors, such as $\mathrm{D} - a$, so that it becomes

$$(\mathrm{D} - a_1)(\mathrm{D} - a_2) \ldots\ldots (\mathrm{D} - a_x)\, u_x = \mathrm{X}.$$

Then taking the inverse operation of $\mathrm{D} - a$,

$$\mathrm{F}'(\mathrm{D})\, u_x = (\mathrm{D} - a_1)^{-1} \mathrm{X} = a_1^{x-1} \Sigma (\mathrm{X} a_1^{-x}),$$

representing the product of the $n - 1$ binomial factors by $\mathrm{F}'(\mathrm{D})$, and omitting the arbitrary constant, as included in the sign of integration. Again, performing the inverse operation of $\mathrm{D} - a_2$, we have

$$\mathrm{F}''(\mathrm{D})\,u_x = a_2^{x-1}\,\Sigma\left\{a_1^{x-1}\,a_2^{-x}\,\Sigma\,(\mathrm{X}a_1^{-x})\right\}$$
$$= \frac{a_2^{x}}{a_1 a_2}\,\Sigma\left\{\left(\frac{a_1}{a_2}\right)^{x}\,\Sigma\,(\mathrm{X}a_1^{-x})\right\},$$

and integrating this by parts,

$$\mathrm{F}''(\mathrm{D})\,u_x = \frac{a_1^{x-1}}{a_1 - a_2}\,\Sigma\,(\mathrm{X}a_1^{-x}) + \frac{a_2^{x-1}}{a_2 - a_1}\,\Sigma\,(\mathrm{X}a_2^{-x}).$$

Proceeding in the same manner with the next factor,

$$\mathrm{F}'''(\mathrm{D})\,u_x = \frac{a_3^{x-1}}{a_1 - a_2}\,\Sigma\left\{a_1^{x-1}a_3^{-x}\,\Sigma\,(\mathrm{X}a_1^{-x})\right\}$$
$$+ \frac{a_3^{x-1}}{a_2 - a_1}\,\Sigma\left\{a_2^{x-1}a_3^{-x}\,\Sigma\,(\mathrm{X}a_2^{-x})\right\}$$
$$= \frac{a_1^{x-1}}{(a_1 - a_2)(a_1 - a_3)}\,\Sigma\,(\mathrm{X}a_1^{-x}) + \frac{a_2^{x-1}}{(a_2 - a_1)(a_2 - a_3)}\,\Sigma\,(\mathrm{X}a_2^{-x})$$
$$+ \frac{a_3^{x-1}}{(a_3 - a_1)(a_3 - a_2)}\,\Sigma\,(\mathrm{X}a_3^{-x}),$$

after integration by parts and reduction. And in the same way we may proceed till all the binomial factors of operation are exhausted. It is obvious from this, how close an analogy this result bears with the corresponding one in Differential Equations.

If there be n equal roots, the theorem given above enables us to arrive directly at the solution. For in this case we should have

$$u_x = (\mathrm{D} - a)^{-n}\,\mathrm{X} = a^{x-n}\,\Sigma^n\,(\mathrm{X}a^{-x}),$$

or introducing the arbitrary constants, which were before neglected,

$$u_x = a^{x-n}\,\Sigma^n\,(\mathrm{X}a^{-x}) + a^{x-n}\,(\mathrm{C} + \mathrm{C}_1 x + \mathrm{C}_2 x^2 + \ldots\ldots + \mathrm{C}_n x^n).$$

Linear equations of the first degree, with variable coefficients, may be easily converted into equations with constant coefficients, by a change analogous to the change of the independent variable in the Differential Calculus. For let

$$u_{x+1} + a\mathrm{P}_x u_x = \mathrm{X}$$

be the given equation, where P_x is a function of x. We may assume $\mathrm{P}_x = \dfrac{\mathrm{Q}_{x+1}}{\mathrm{Q}_x}$, so that dividing by Q_{x+1} the equation becomes

$$\frac{u_{x+1}}{\mathrm{Q}_{x+1}} + a\,\frac{u_x}{\mathrm{Q}_x} = \frac{\mathrm{X}}{\mathrm{Q}_x};$$

which may be solved as a linear equation in $\dfrac{u_x}{\mathrm{Q}_x}$, from which, when found, we may determine u_x. The form of Q_{x+1} is evidently

$$\mathrm{P}_1\mathrm{P}_2 \ldots\ldots \mathrm{P}_x.$$

The same method may be sometimes applied to equations of a higher order. If, for instance, we have the equation

$$u_{x+2} + a\phi(x+1)\,u_{x+1} + b\phi(x)\,\phi(x+1)\,u_x = c.$$

Let us represent the continued product $\phi(1).\phi(2)\ldots\phi(x+1)$ by P_{x+1}. Then

$$\phi(x)\,\phi(x+1) = \frac{P_{x+1}}{P_{x-1}} \text{ and } \phi(x+1) = \frac{P_{x+1}}{P_x}.$$

Dividing by P_{x+1} we get

$$\frac{u_{x+2}}{P_{x+1}} + a\,\frac{u_{x+1}}{P_x} + b\,\frac{u_x}{P_{x-1}} = \frac{C}{P_{x+1}},$$

which may be solved as an equation in $\frac{u_x}{P_{x-1}}$, and so the value of u_x determined.

It will be seen that the method here laid down of integrating equations of differences, possesses an advantage over the common one of not requiring any assumption of a form of solution; but this advantage is even more displayed in solving equations of mixed differences. In these the two symbols d and Δ are involved at the same time; but as they are the characteristics of independent operations, the same principles which were applied to each separately will hold when they are combined. Without discussing the general equations of mixed differences, we shall proceed to a few of the more interesting examples. Let us have the equation

$$\frac{d}{dx}\Delta y + a\,\frac{dy}{dx} + b\Delta y + aby = X.$$

By separating the symbols this may be put under the form

$$\left(\frac{d}{dx} + b\right)(\Delta + a)\,y = X.$$

We shall obtain a different result according as we operate first with the one or the other factor; taking then the differential factor first, we have

$$(\Delta + a)\,y = \varepsilon^{-bx}\int X\varepsilon^{bx}dx + C\varepsilon^{-bx}.$$

$$\text{Now } \Delta + a = 1 + \Delta - (1 - a) = D - (1 - a).$$

Therefore, performing the inverse operation by the theorem given above, page 55, we obtain

$$y = (1-a)^{x-1}\,\Sigma\{(1-a)^{-x}\varepsilon^{-bx}\int(X\varepsilon^{bx}dx)\}$$
$$+ C(1-a)^{x-1}\,\Sigma\{\varepsilon^{-bx}(1-a)^{-x}\} + (1-a)^x\phi(\sin 2\pi x,\ \cos 2\pi x),$$

the last term being the complementary function arising from the operation $\{D - (1 - a)\}^{-1}$.

And performing the operation indicated in the second term, we have

$$y = (1-a)^{x-1}\,\Sigma\{(1-a)^{-x}\varepsilon^{-bx}\int(X\varepsilon^{bx}dx)\}$$
$$+ C_1\varepsilon^{-bx} + (1-a)^x\phi(\sin 2\pi x,\ \cos 2\pi x).$$

If we begin with the other factor we have

$$\left(\frac{d}{dx} + b\right) y = (1-a)^{x-1}\,\Sigma\{\mathrm{X}\,(1-a)^{-x}\} + (1-a)^x\,\phi$$

$$\text{and } y = \varepsilon^{-bx}\int dx\,\{\varepsilon^{bx}\,(1-a)^{x-1}\,\Sigma\,(\mathrm{X}\,(1-a)^{-x})\} + \varepsilon^{-bx}\int dx\,\{\varepsilon^{bx}\,(1-a)^x\,\phi\} + \mathrm{C}\varepsilon^{-bx},$$

where ϕ is put for $\phi\,(\sin 2\pi x,\ \cos 2\pi x)$.

The next example we shall take is one given by Paoli, where two variables are involved,

$$u_{x+1,\,y} - \frac{d}{dy}\,u_{x,\,y} = \mathrm{P}_{x,\,y},$$

which may be put under the form

$$\left(\mathrm{D} - \frac{d}{dy}\right) u_{x,\,y} = \mathrm{P}_{x,\,y},$$

where the D refers to the x only.

Then, following the same principle as that pursued by Mr. Greatheed in the *Philosophical Magazine* for September 1837, that is, considering the characteristic $\frac{d}{dy}$ as an independent quantity, we have

$$u_{x,\,y} = \left(\frac{d}{dy}\right)^{x-1} \Sigma \left(\frac{d}{dy}\right)^{-x} \mathrm{P}_{x,\,y} + \left(\frac{d}{dy}\right)^x \phi\,(y),$$

an arbitrary function of y being substituted for a constant, as the operations are partial. This may evidently be put under the form

$$u_{x,\,y} = \frac{d^x\,(\Sigma\int^{x+1} \mathrm{P}_{x,\,y}\,dy^{x+1})}{dy^x} + \frac{d^x\,\phi\,(y)}{dy^x}.$$

If $\mathrm{P}_{x,\,y} = 0$, the result is the same as that given by Sir John Herschel, in p. 38 of his *Examples*. He likewise gives the equation

$$u_{x+2,\,y} - a\left(\frac{d}{dy}\right) u_{x+1,\,y} + b\left(\frac{d^2}{dy^2}\right) u_{x,\,y} = 0,$$

which may be put under the form

$$\left(\mathrm{D}^2 - a\mathrm{D}\,\frac{d}{dy} + b\,\frac{d^2}{dy^2}\right) u_{x,\,y} = 0,$$

where D affects x only. Or, putting it into the form of factors,

$$\left(\mathrm{D} - m\,\frac{d}{dy}\right)\left(\mathrm{D} - n\,\frac{d}{dy}\right) u_{x,\,y} = 0,$$

where m, n are the roots of the equation

$$z^2 - az + b = 0.$$

Then $\left(\mathrm{D} - n\,\frac{d}{dy}\right) u_{x,\,y} = m^{x-1}\cdot\left(\frac{d}{dy}\right)^{x-1}\cdot\phi\,(y),$

and $u_{x,y} = n^{x-1}.\left(\frac{d}{dy}\right)^{x-1} \Sigma\left\{m^{x-1}.n^{-x}.\left(\frac{d}{dy}\right)^{x-1}.\left(\frac{d}{dy}\right)^{-x}.\phi(y)\right\}$

$$+ n^{x-1}\left(\frac{d}{dy}\right)^{x-1}\phi_1(y);$$

whence $u_{x,y} = \frac{m^{x-1}d^{x-1}\phi_2(y)}{dy^{x-1}} + \frac{n^{x-1}d^{x-1}\phi_1(y)}{dy^{x-1}}$,

as the functions of y are quite arbitrary.

In the same manner we might integrate this equation, if the second side were some function of x; and also equations of the form

$$\Delta^2 u_{x,y} - a\Delta\frac{d}{dy}u_{x,y} + b\frac{d^2}{dy^2}u_{x,y} = \mathrm{X},$$

where Δ affects x only. But it is needless to dwell on these, and we shall therefore proceed to the integration of an equation, in which the advantages of this method are very conspicuous. Mr. Airy, on the hypothesis that the distances of the particles of the luminiferous ether are not infinitely small compared with their sphere of action, has given the following equation for determining the disturbance of a particle:

$$\frac{d^2}{dt^2}u_{x,t} = \frac{a^2}{h^2}\Delta^2 u_{x-h,t},$$

where Δ affects x only. This may be put under the form

$$\left(\frac{d^2}{dt^2} - \frac{a^2}{h^2}\frac{\Delta^2}{1+\Delta}\right)u_{x,t} = 0.$$

Following the same method as before, by splitting the operating factor into two, and integrating with each separately, we obtain

$$u_{x,t} = \varepsilon^{\frac{a}{h}t\frac{\Delta}{\sqrt{1+\Delta}}}\phi(x) + \varepsilon^{-\frac{a}{h}t\frac{\Delta}{\sqrt{1+\Delta}}}\psi(x),$$

where $\phi(x)$ and $\psi(x)$ are arbitrary functions of x. To reduce this to a more intelligible form, we may avail ourselves of Fourier's formula for the transformation of functions. From it we have

$$\pi\phi(x) = \int_{-\infty}^{+\infty} da\,\phi(a)\int_0^{\infty} dp\,\cos p(x-a).$$

Substituting this form,

$$\pi u_{x,t} = \int_{-\infty}^{+\infty} da\,\phi(a)\int_0^{\infty} dp\,\varepsilon^{\frac{a}{h}t\frac{\Delta}{\sqrt{1+\Delta}}}\cos p(x-a)$$

$$+\int_{-\infty}^{+\infty} da\,\psi(a)\int_0^{\infty} dp\,\varepsilon^{-\frac{a}{h}t\frac{\Delta}{\sqrt{1+\Delta}}}\cos p(x-a).$$

We must now expand the exponential, and operate with each term separately on the cosine. When expanded it becomes

$$\left(1 + \frac{a}{h}t\frac{\Delta}{\sqrt{1+\Delta}} + \frac{a^2h}{h^2 1.2}t^2\frac{\Delta^2}{1+\Delta} + \&c.\right)\cos p(x-a).$$

Now

$$\frac{\Delta}{\sqrt{1+\Delta}} f(x) = \left(\sqrt{1+\Delta} - \frac{1}{\sqrt{1+\Delta}}\right) f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right);$$

therefore $$\frac{\Delta}{\sqrt{1+\Delta}} \cos p\,(x-a) = -2 \sin \frac{ph}{2} \sin p\,(x-a),$$

$$\frac{\Delta^2}{1+\Delta} \cos p\,(x-a) = -\left(2 \sin \frac{ph}{2}\right)^2 \cos p\,(x-a),$$

and so on for every repetition of the operation, so that the series becomes

$$\left\{1 - \frac{a^2.h^2}{h^2.1.2}\left(2 \sin \frac{ph}{2}\right)^2 + \frac{a^4.t^4}{h^4.1.2.3.4}\left(2 \sin \frac{ph}{2}\right)^4 - \&c.\right\} \times$$

$$\times \cos p\,(x-a)$$

$$-\left\{\frac{a}{h}\frac{t}{1}\left(2 \sin \frac{ph}{2}\right) - \frac{a^3.t^3}{h^3.1.2.3}\left(2 \sin \frac{ph}{2}\right)^3 + \&c.\right\} \sin p\,(x-a)$$

$$= \cos\left(\frac{2at}{h} \sin \frac{ph}{2}\right) \cos p\,(x-a) - \sin\left(\frac{2at}{h} \sin \frac{ph}{2}\right) \sin p\,(x-a).$$

Similarly for the other term we shall obtain

$$\cos\left(\frac{2at}{h} \sin \frac{ph}{2}\right) \cos p\,(x-a) + \sin\left(\frac{2at}{h} \sin \frac{ph}{2}\right) \sin p\,(x-a);$$

and if these expressions be substituted in the expression for $u_{x,t}$, we shall obtain a result free from the symbol Δ: but a slight change in the form of the function will reduce it to a somewhat better form; for if we make

$$f(a) = \psi(a) + \phi(a) \text{ and } \mathrm{F}(a) = \psi(a) - \phi(a),$$

the expression becomes

$$\pi u_{x,t} = \int_{-\infty}^{+\infty} da f(a) \int_0^{\infty} dp \cos\left(\frac{2at}{h} \sin \frac{ph}{2}\right) \cos p\,(x-a)$$

$$+ \int_{-\infty}^{+\infty} da\, \mathrm{F}(a) \int_0^{\infty} dp \sin\left(\frac{2at}{h} \sin \frac{ph}{2}\right) \sin p\,(x-a),$$

which is the general solution of the equation. The form of the arbitrary functions are easily determined from the initial circumstances. For if in the last expression, or in the original integral, we make $t = 0$, we find

$$u_{x,0} = f(x),$$

which determines the form of the function f from the initial state of the particles. If we differentiate the expression for $u_{x,t}$ with regard to t, and then make $t = 0$, there results

$$\frac{du_{x,0}}{dt} = \frac{a}{h}\left\{\mathrm{F}\left(x-\frac{h}{2}\right) - \mathrm{F}\left(x+\frac{h}{2}\right)\right\},$$

which determines the form of the function F from the initial velocity.

Mr. Airy has found a particular integral of this equation by assuming as a form of solution

$$u = \mathrm{A} \sin \frac{2\pi}{\lambda} (vt - x + a),$$

and then determining v. But his result may be arrived at, merely by assuming the form of the arbitrary function to be that used by Taylor in the solution of the problem of vibrating chords, and which is usually taken as the form of the function in the undulatory theory, and is for that reason well adapted for comparing the results. Let then

$$\phi(x) = \sin \frac{2\pi}{\lambda} (x + a), \quad \psi(x) = 0.$$

Effecting the operations indicated in the original integral by the same method as that used in reducing Fourier's function, we obtain

$$u = \sin \frac{2\pi}{\lambda} \left\{ x + \frac{a\lambda}{\pi} \sin \left(\frac{\pi h}{\lambda} \right) t + a \right\}.$$

This expression gives us for the velocity of propagation of the wave, which is the coefficient of t,

$$v = \frac{a\lambda}{\pi} \sin \frac{\pi h}{\lambda},$$

which is no longer independent of λ, and therefore the velocity will be different for different values of λ; and as the index of refraction is inversely as the velocity, it will be nearly inversely as λ, and will thus be greater for violet than red light. This result then, as far as it goes, accounts for the phenomena of dispersion.

Trin. Coll. D. F. G.

III.—DEMONSTRATIONS OF SOME PROPERTIES OF THE CONIC SECTIONS.

1. The position of the circle of curvature at any point of a Conic Section, may be readily determined by the following construction. Describe a circle touching the curve at the given point and cutting it in two others, then the chord in the conic section which passes through the given point and is parallel to the line joining the two points of section, is a chord of the circle of curvature at the given point.

Taking the origin at the point in the curve, the equation to the conic section is

$$y^2 + \mathrm{B}xy + \mathrm{C}x^2 + \mathrm{D}y + \mathrm{E}x = 0.$$

If we make the normal the axis of x and the tangent that of y, when $x = 0$ the two values of y must also $= 0$, and therefore $D = 0$, which reduces the equation to

$$y^2 + Bxy + Cx^2 + Ex = 0.$$

The equation to a circle touching the curve at the given point, that is, being also a tangent to the axis of y at that point, is

$$y^2 + x^2 + E'x = 0.$$

At the intersection of the curves we may combine the equations in any manner; subtracting them we have

$$Bxy + (C - 1)x^2 + (E - E')x = 0,$$

which gives

$$x = 0, \quad By + (C - 1)x + E - E' = 0.$$

The first of these equations gives the origin, the second gives a relation between the coordinates of each of the points of section; and as it is linear it is the equation to the straight line joining them. Now the angle which this line makes with the axis of x is determined by the ratio $\frac{1 - C}{B}$; and as this is independent of the circle, the line will remain parallel to one position, as the circle varies in size. But when the circle becomes the circle of curvature, one of the points of intersection coincides with the point of contact, and the line joining the points of intersection, which remains parallel to one position, must pass through the point of contact. Therefore, if from this point a line be drawn parallel to a known position of the line of intersection, it will pass through the point in which the circle of curvature cuts the curve, and will thus be a chord both in the curve itself and in the circle of curvature. Knowing now one chord of the circle of curvature, and the position of its diameter, which coincides with that of the normal, we can determine the circle altogether.

2. The following is another and a very curious method of determining the centre of curvature in the Conic Sections. It was first given by Keill, but seems to have been rather strangely neglected by the subsequent writers on this subject.

From any point P (fig. 1) in the curve draw the normal PN cutting the axis in N; at the point N draw NQ perpendicular to the normal, and meeting the focal chord through P in Q. From Q draw QO perpendicular to the focal chord and meeting the normal in O; then O is the centre of the circle of curvature at the point P. Draw FY from the focus perpendicular to the tangent, and let

$$FP = r, \quad FY = p, \quad PN = N.$$

In the right-angled triangle PQO, we have plainly

$$PO.PN = PQ^2.$$

But from the similar triangles QPN, FPY,

$$PQ.FY = FP.PN;$$

$$\text{therefore } PQ^2 = N^2 \frac{r^2}{p^2}, \text{ and } PO = N \frac{r^2}{p^2}.$$

Also in a conic section we have, if R be the radius of curvature,

$$R = \frac{N^3}{m^2} = \text{also } \frac{mr^3}{p^3},$$

where m is half the *latus rectum*.

From which we have $\frac{r^2}{p^2} = \frac{N^2}{m^2}$, and therefore

$$PO = \frac{N^3}{m^2} = R,$$

and O is the centre of the circle of curvature at P.

3. In the *Cambridge Transactions*, Vol. 3, Mr. Morton has demonstrated a number of curious properties of the Conic Sections in relation to the generating cone; but he does not seem to have noticed the following one. If a sphere be described round the vertex of a cone as centre, the *latera recta* of all sections of the cone made by planes touching the sphere are equal. Taking the vertex of the cone as the origin, and the axis of the cone as the axis of x, the equation to the cone will be

$$x^2 + y^2 = m^2 z^2.$$

And if we suppose the cutting plane to be perpendicular to the plane of xz, its equation will be

$$z \cos a + x \sin a = r;$$

where r is the radius of the sphere, and a the angle which the perpendicular from the origin on the plane makes with the axis of z. Eliminating z between the two equations, we get

$$x^2 (\cos^2 a - m^2 \sin^2 a) + y^2 \cos^2 a + 2m^2 rx \sin a = m^2 r^2,$$

which is the equation to the projection of the section on the plane of xy. This equation, which is that to an ellipse, is not referred to its centre, but if we so refer it, it becomes

$$x^2 (\cos^2 a - m^2 \sin^2 a)^2 + y^2 \cos^2 a (\cos^2 a - m^2 \sin^2 a) = m^2 r^2 . \cos^2 a.$$

Now, if a', b' be the axes of the projection,

$$a'^2 = \frac{m^2 r^2 \cos^2 a}{(\cos^2 a - m^2 \sin^2 a)^2}, \quad b'^2 = \frac{m^2 r^2}{\cos^2 a - m^2 \sin^2 a}.$$

If a, b be the axes of the section, as the cutting plane is perpendicular to the plane of xz, and makes an angle a with the plane of xy,

$$a' = a \cos a, \quad b' = b.$$

And for the *latus rectum*,

$$\frac{2b^2}{a} = 2mr,$$

which being independent of a, is the same for all sections for which r is the same; that is, for all those which are made by planes touching the sphere.

From this it appears, that the *latus rectum* is equal to the diameter of the sphere multiplied by the tangent of half the vertical angle of the cone.

D.

IV.—ON THE MOON'S ANNUAL AND SECULAR INEQUALITIES.

THE following is a simple method of obtaining the expressions for the Moon's annual and secular equations, and which may be applied to determine those parts of the Moon's inequalities in longitude, which depend on the central disturbing force, and are independent of the eccentricity of the Moon's orbit and the square of the disturbing force.

In this investigation the Moon is supposed to be moving in a circle, the radius of which is variable. The equable description of areas is independent of the magnitude of the central force. Hence

$$h = (\mu a)^{\frac{1}{2}} \text{ is constant, } \mu \text{ and } a \text{ varying.}$$

Let $\delta\mu$, δa represent the *finite* variations of μ and a at any time. We suppose their squares to be neglected in this investigation, and hence they may be treated as if they were infinitesimal.

Differentiating the logarithm of the expression for h,

$$0 = \frac{\delta\mu}{\mu} + \frac{\delta a}{a} \ldots\ldots\ldots\ldots (1);$$

$$\text{also } \frac{d\theta}{dt} = \sqrt{\frac{\mu}{a^3}};$$

$$\text{therefore } \frac{\delta d\theta}{d\theta} = \frac{1}{2}\cdot\frac{\delta\mu}{\mu} - \frac{3}{2}\cdot\frac{\delta a}{a} \ldots\ldots (2):$$

and substituting the value of $\frac{\delta a}{a}$ from equation (1), and putting $d\delta\theta$ for $(\delta d\theta)$,

$$\frac{d\delta\theta}{d\theta} = 2\frac{\delta\mu}{\mu},$$

$$\delta\theta = 2\int\frac{\delta\mu}{\mu}\,d\theta \ldots\ldots (3),$$

which will give the Moon's inequality in longitude.

The central force is by hypothesis $\frac{\mu + \delta\mu}{a^2}$; in the case of the Moon disturbed by the Sun, this is equal to $\frac{\mu}{a^2} - \frac{1}{2}\frac{m'a}{r'^3}$, where m', r' are the Sun's mass and distance.

$$\text{Hence } \frac{\delta\mu}{\mu} = -\frac{1}{2}\frac{m'a^3}{\mu r'^3}.$$

As the Sun describes an elliptic path about the Earth,

$$\frac{1}{r'} = \frac{1 + e'\cos(\theta' - \omega)}{a'(1 - e'^2)}.$$

Let m be the ratio of mean motions of the Sun and Moon, β the Sun's longitude when the Moon's longitude is 0,

$$\theta' = m\theta + \beta + \text{terms multiplied by } e'.$$

Hence, as far as the first power of e',

$$\frac{1}{r'} = \frac{1 + e'\cos(m\theta + \beta - \omega')}{a'};$$

substituting this value, and putting m^2 for $\frac{m'}{\mu}\cdot\frac{a^3}{a'^3}$,

$$\frac{\delta\mu}{\mu} = -\frac{m^2}{2} - \frac{3}{2}m^2e'\cos(m\theta + \beta - \omega').$$

The effect of the first term of this expression is to increase the Moon's periodic time by a constant quantity, which is in fact included in the mean motion determined from observation. The second term will give the inequality depending on the place of the Sun and the eccentricity of its orbit. Therefore, confining $\delta\theta$ to this part,

$$\begin{aligned}\delta\theta &= -3m^2e'\int\cos(m\theta + \beta - \omega')\,d\theta,\\ &= -3me'\sin(m\theta + \beta - \omega'),\\ &= -3me'\sin(\text{Sun's mean anomaly});\end{aligned}$$

which is the expression for the annual equation. Vide *Airy*, Lunar Theory, No. 59.

The effect of that part of the Sun's attraction which is independent of the Moon's place in her orbit, is to diminish the Earth's attraction by the quantity

$$\frac{1}{2}\frac{m'a}{r'^3}.$$

This depends on the Sun's position in its orbit.

Its mean value will be found by adding its values taken at every instant during a complete revolution of the Sun, *i. e.* a year, and dividing by the number of instants; this amounts to integrating the quantity, multiplied by the element of time, for a year, and dividing by the length of a year.

Let T represent the length of a year.

$$\text{The mean value of } \tfrac{1}{2}\frac{m'a}{r'^3} = \frac{1}{\mathrm{T}}\cdot\tfrac{1}{2}m'a\int_0^{\mathrm{T}}\frac{dt}{r'^3},$$

$$r'^2 dj' = \sqrt{m'a'(1-e'^2)}\,dt,$$

$$\mathrm{T} = 2\pi a'^{\frac{3}{2}} m'^{-\frac{1}{2}}.$$

Hence mean value

$$= \frac{m'a}{r}\cdot\frac{\sqrt{m'}}{2\pi a'^{\frac{3}{2}}}\cdot\frac{1}{\sqrt{m'a'(1-e'^2)}}\cdot\int_0^{2\pi}\frac{d\theta'}{r'},$$

$$= \frac{1}{4\pi}\cdot\frac{m'a}{a'^2}\cdot\frac{1}{\sqrt{1-e'^2}}\int_0^{2\pi}\frac{1+e'\cos(\theta'-\omega')}{a'(1-e'^2)}\,d\theta',$$

$$= \tfrac{1}{2}\cdot\frac{m'a}{a'^3}\cdot\frac{1}{(1-e'^2)^{\frac{3}{2}}},$$

$$= \tfrac{1}{2}\frac{m^2}{(1-e'^2)^{\frac{3}{2}}}\cdot\frac{\mu}{a^2}.$$

Neither the Sun's mean distance, nor the length of the year which depends on it, alter sensibly. But in consequence of the action of the planets, the eccentricity of its orbit e' has been constantly diminishing for centuries. The expression we have just found shows, that in consequence of this the value of the Sun's disturbing force is constantly diminishing, or the Earth's attraction increasing; and hence arises that inequality celebrated in the history of Astronomy as the secular acceleration of the Moon's mean motion.

To find its amount, suppose E the eccentricity of the Sun's orbit at any epoch,

$$2\frac{\partial\mu}{\mu} = -\frac{m^2}{(1-e'^2)^{\frac{3}{2}}},$$

$$= -m^2 - \frac{3}{2}m^2e'^2 + \&c.$$

$$= -m^2 - \frac{3}{2}m^2\mathrm{E}^2 + \frac{3}{2}m^2(\mathrm{E}^2 - e'^2) + \&c.$$

The effect of the first two terms is included in the Moon's mean motion, as determined by observation at the epoch in question; the inequality in longitude at any other time, depending on the diminution of e', will be

$$\partial\theta = \frac{3}{2}m^2\int(\mathrm{E}^2 - e'^2)\,d\theta,$$

which is the equation given in the *Mécanique Céleste*, liv. vii. No. 15.

Let E′ be the annual diminution of e', and let the year be taken as the unit of time;

$$e' = \mathrm{E} - \mathrm{E}'t + \&c.; \quad \therefore\ \mathrm{E}^2 - e'^2 = 2\mathrm{E}\mathrm{E}'t \text{ nearly,}$$

$$md\theta = 2\pi dt; \text{ therefore}$$

$$\delta\theta = 3m\pi\mathrm{E}\mathrm{E}'\int 2t\,dt,$$

$$= 3m\pi\mathrm{E}\mathrm{E}'t^2,$$

$$= 3m\pi(\text{ecc}^{\text{y}}.\text{ of Sun's orb.})(\text{ann. var. of ecc}^{\text{y}}.)(\text{sq. of n}^{\text{o}}.\text{ of yrs.}).$$

H. T.

V.—ON THE EXPANSION OF A FUNCTION OF A BINOMIAL.

Though Taylor's theorem naturally presents itself at the very entrance to the Differential Calculus, many points relating to it are imperfectly understood even by the best mathematicians. It does not seem to have occurred to any one, that the particular expansions of $f(a + h)$ in fractional or negative powers of h, to which we are obliged to have recourse where Taylor's series fails when x becomes a, are derivable from another general expansion of $f(x + h)$. This we shall show to be the case. It seems, indeed, that many persons, misled by Lagrange's proposition, "the expansion of $f(x + h)$ cannot contain any fractional or negative powers of h unless x have a particular value," think that there is no such other general expansion; but the above proposition ought to be enounced thus, "that expansion of $f(x + h)$ *whose first term is* $f(x)$, cannot contain any fractional or negative powers of h, *and cannot fail* unless x have a particular value;" for the propositions, "the first term of the expansion is $f(x)$," and "the expansion cannot contain fractional or negative powers of h," are reciprocally dependent on one another, as will be seen. Professor Peacock, in his Report referred to in page 11 of our last Number, has employed general differentiation to express the general term of the expansion of $f(x + h)$; but in all other respects we believe that the contents of this Article are new.

2. Let the following general series be assumed for $f(x + h)$,

$$\mathrm{A}h^{\alpha} + \mathrm{A}'h^{\alpha'} + \mathrm{A}''h^{\alpha''} + \ldots\ldots\ldots,$$

where the indices α, α', α'' are supposed to be in order of magnitude, either increasing or decreasing. Equating the first differential coefficients with respect to x and h,

$$\left.\begin{aligned}&\frac{dA}{dx}h^{\alpha} + \frac{dA'}{dx}h^{\alpha'} + \frac{dA''}{dx}h^{\alpha''} + \ldots\\ &= \alpha A h^{\alpha-1} + \alpha' A' h^{\alpha'-1} + \alpha'' A'' h^{\alpha''-1} + \ldots\end{aligned}\right\} \ldots\ldots\ldots (1).$$

One way by which this equation may be satisfied is, by making $\alpha = 0$; this will lead to the known series of Taylor. But there is another way by which equation (1) may be satisfied; this is, by making $\frac{dA}{dx}$ equal to zero. Others of the differential coefficients may or may not vanish; we will suppose first, for the sake of simplicity, that they do not. Comparing the other terms with respect both to the indices and coefficients,

$$\alpha' = \alpha - 1, \quad \alpha'' = \alpha' - 1, \text{ \&c.}$$

$$\frac{dA'}{dx} = \alpha A, \quad \frac{dA''}{dx} = \alpha' A', \text{ \&c.;}$$

and since A is a constant $= C_0$,

$$A' = \int \alpha A\, dx = \frac{\alpha}{1} C_0 x + C_1,$$

$$A'' = \int (\alpha - 1)\, A' dx = \frac{\alpha(\alpha-1)}{1.2} C_0 x^2 + \frac{\alpha - 1}{1} C_1 x + C_2, \text{ \&c.}$$

Hence

$$f(x + h) = C_0 h^{\alpha} + \left(\frac{\alpha}{1} C_0 x + C_1\right) h^{\alpha-1}$$

$$+ \left(\frac{\alpha(\alpha-1)}{1.2} C_0 x^2 + \frac{\alpha - 1}{1} C_1 x + C_0\right) h^{\alpha-2} + \ldots\ldots (2).$$

3. It is only in a particular form of $f(x)$ that $f(x + h)$ can be expanded in this series. To determine that form, put $x = 0$ in the last equation, whence

$$f(h) = C_0 h^{\alpha} + C_1 h^{\alpha-1} + C_2 h^{\alpha-2} + \ldots\ldots,$$

and changing h into x,

$$f(x) = C_0 x^{\alpha} + C_1 x^{\alpha-1} + C_2 x^{\alpha-2} + \ldots\ldots\ldots (3).$$

It follows then, that when $f(x)$ is of, or can be expanded in the form (3), which includes all the simplest functions for which Taylor's series can fail, there exists another general development of $f(x + h)$ besides Taylor's series. In fact, the series (2) is the same as would result from changing x into $x + h$ in each term of (3), and developing each term by the binomial theorem in descending powers of h. It is evident that the series (2) cannot fail for any value of x. If all the indices of x in (3) be positive integers, the series (2) will be identical with Taylor's (which will in this case terminate) written in the contrary order.

4. The coefficients of the several powers of h in (2) may be expressed in terms of the general differential coefficients of $f(x)$ with respect to x. Putting $f(x) = u$, and using the formula (P) in page 20 of our last Number, we evidently have from (3),

$$\frac{d^{a-r}u}{dx^{a-r}} = \frac{\mathrm{P}(a)}{\mathrm{P}(r)}\mathrm{C}_0x^r + \frac{\mathrm{P}(a-1)}{\mathrm{P}(r-1)}\mathrm{C}_1x^{r-1} + \ldots\ldots$$

$$= \frac{a(a-1)\ldots(a-r+1)\,\mathrm{P}(a-r)}{1.2\ldots r}\mathrm{C}_0x^r$$

$$+ \frac{(a-1)(a-2)\ldots(a-r+1)\,\mathrm{P}(a-r)}{1.2\ldots(r-1)}\mathrm{C}_1x^{r-1} + \ldots$$

As was proved in the article on General Differentiation, the terms in the preceding differential coefficient which contain negative powers of x (if there be any such) are finite, but, unless $a-r$ be a positive integer, $\mathrm{P}(a-r)$ is infinite; hence in dividing by $\mathrm{P}(a-r)$ the terms involving negative powers of x disappear, and we have

$$\frac{1}{\mathrm{P}(a-r)}\frac{d^{a-r}u}{dx^{a-r}} = \frac{a(a-1)\ldots(a-r+1)}{1.2\ldots r}\mathrm{C}_0x^r$$

$$+ \frac{(a-1)(a-2)\ldots(a-r+1)}{1.2\ldots(r-1)}\mathrm{C}_1x^{r-1} + \ldots + \mathrm{C}_r,$$

which is identical with the coefficient of h^{a-r} in equation (2); hence when $f(x)$ is of the form (3), $f(x+h)$ may be expanded in the series

$$\frac{h^a}{\mathrm{P}(a)}\frac{d^au}{dx^a} + \frac{h^{a-1}}{\mathrm{P}(a-1)}\frac{d^{a-1}u}{dx^{a-1}} + \frac{h^{a-2}}{\mathrm{P}(a-2)}\frac{d^{a-2}u}{dx^{a-2}} + \ldots\ldots (4).$$

When a is a positive integer, the terms involving negative powers of x would not disappear naturally from those differential coefficients whose index is positive, but they must be rejected, in order that the expansion may agree with (2).

5. The following examples may serve to explain the subject.

Let $u = x\sqrt{a-x}$: the expansion of this in descending powers of x is

$$\sqrt{-1}\left(x^{\frac{3}{2}} - \frac{1}{2}ax^{\frac{1}{2}} - \frac{1.1}{2.4}a^2x^{-\frac{1}{2}} - \frac{1.1.3}{2.4.6}a^3x^{-\frac{3}{2}} + \ldots\ldots\right);$$

which is of the form (3), and we find in this case $a = \frac{3}{2}$. Also from this expansion we obtain

$$\frac{d^{\frac{3}{2}}u}{dx^{\frac{3}{2}}} = \sqrt{-1}\left\{\mathrm{P}\left(\frac{3}{2}\right) - \frac{1}{2}\frac{\mathrm{P}\left(-\frac{1}{2}\right)}{\mathrm{P}(-1)}ax^{-1} - \frac{1.1}{2.4}\frac{\mathrm{P}\left(-\frac{1}{2}\right)}{\mathrm{P}(-2)}a^2x^{-2} - \ldots\right\},$$

$$\frac{d^{\frac{1}{2}}u}{dx^{\frac{1}{2}}} = \sqrt{-1}\left\{\frac{\mathrm{P}\left(\frac{3}{2}\right)}{\mathrm{P}(1)}x - \frac{1}{2}\mathrm{P}\left(\frac{1}{2}\right)a - \frac{1.1}{2.4}\frac{\mathrm{P}\left(-\frac{1}{2}\right)}{\mathrm{P}(-1)}a^2x^{-1} - \ldots\right\},$$

$$\frac{d^{-\frac{1}{2}}u}{dx^{-\frac{1}{2}}} = \sqrt{-1}\left\{\frac{\mathrm{P}\left(\frac{3}{2}\right)}{\mathrm{P}(2)}x^2 - \frac{1}{2}\frac{\mathrm{P}\left(\frac{1}{2}\right)}{\mathrm{P}(1)}ax - \frac{1.1}{2.4}\mathrm{P}\left(-\frac{1}{2}\right)a^2 - \ldots\right\},$$

&c.

Substituting these values in the series (4), observing the relation $\mathrm{P}(n) = n\mathrm{P}(n-1)$, and that $\mathrm{P}(-1) = \int x^{-1}dx = \frac{1}{0}$, we find

$$(x+h)\sqrt{a-x-h} =$$

$$\sqrt{-1}\left\{h^{\frac{3}{2}} + \left(\frac{3}{2}x - \frac{1}{2}a\right)h^{\frac{1}{2}} + \left(\frac{3.1}{2.4}x^2 - \frac{1}{2}\cdot\frac{1}{2}ax - \frac{1.1}{2.4}a^2\right)h^{-\frac{1}{2}} + \ldots\right\};$$

the $(p+1)^{th}$ term being

$$\sqrt{-1}\left\{\frac{3.1.(-1)\ldots(5-2p)}{2.4.6\ldots\ldots\ldots 2p}x^p - \frac{1}{2}\cdot\frac{1.(-1)\ldots(5-2p)}{2.4\ldots\ldots\ldots(2p-2)}ax^{p-1}\right.$$

$$\left. - \frac{1.1}{2.4}\cdot\frac{(-1)\ldots(5-2p)}{2\ldots\ldots(2p-4)}a^2x^{p-2} - \ldots - \frac{1.1\ldots(3-2p)}{2.4\ldots\ldots\ldots 2p}a^p\right\}h^{\frac{3-2p}{2}};$$

when x receives the particular value a, the terms after the second vanish, and the series becomes

$$\sqrt{-1}\,(h^{\frac{3}{2}} + ah^{\frac{1}{2}}).$$

6. Again, let u or $f(x) = \sqrt{x^2 - a^2}$. Since this is equal to

$$x - \frac{1}{2}\frac{a^2}{x} - \frac{1.1}{2.4}\frac{a^4}{x^3} - \ldots$$

we have $a = 1$,

$$\frac{du}{dx} = 1 + \frac{1}{2}\frac{a^2}{x^2} + \frac{1.1}{2.4}\frac{3a^4}{x^4} + \ldots\ldots,$$

$$\frac{d^{-1}u}{dx^{-1}} = \frac{x^2}{2} - \frac{1}{2}\frac{a^2x^0}{0} + \frac{1.1}{2.4}\frac{a^4}{2.x^2} + \ldots\ldots,$$

$$\frac{d^{-2}u}{dx^{-2}} = \frac{x^3}{2.3} - \frac{1}{2}\frac{a^2x}{0.1} - \frac{1.1}{2.4}\frac{a^4}{2.1.x} - \ldots\ldots,$$

$$\frac{d^{-3}u}{dx^{-3}} = \frac{x^4}{2.3.4} - \frac{1}{2}\frac{a^2x^2}{0.1.2} - \frac{1.1}{2.4}\frac{a^4x^0}{2.1.0} + \ldots\ldots,$$

and since the values of $\mathrm{P}(1)$, $\mathrm{P}(0)$, $\mathrm{P}(-1)$, $\mathrm{P}(-2)$, $\mathrm{P}(-3)$, are 1, $\frac{1}{0}$, $-\frac{1}{1.0}$, $\frac{1}{2.1.0}$ respectively, we have, rejecting negative powers of x from $\frac{du}{dx}$ and u,

$$f(x+h) = h + xh^0 - \frac{1}{2}a^2h^{-1} + \frac{1}{2}a^2xh^{-2} - \left(\frac{1}{2}a^2x^2 + \frac{1.1}{2.4}a^4\right)h^{-3} + \ldots,$$

the $(p+1)^{th}$ term being (provided p be not less than 2)

$$(-1)^{p+1}\left(\frac{1}{2}a^2x^{p-2} + \frac{1.1}{2.4}\cdot\frac{(p-2)(p-3)}{1.2}a^4x^{p-4}\right.$$

$$\left. + \frac{1.1.3}{2.4.6}\cdot\frac{(p-2)(p-3)(p-4)(p-5)}{1\ .\ 2\ .\ 3\ .\ 4}a^6x^{p-6} + \ldots\ldots\right).$$

This will agree with the expansion of $\sqrt{h^2 \pm 2ah}$, when x has the value $\pm a$.

7. Returning to equation (1), let us suppose that several of the terms in its first member, besides $\frac{d\Lambda}{dx}$, vanish, and that the indices of h in these terms are β, γ, &c. By comparing the other terms with respect to their indices and coefficients, we should obtain for the expression of $f(x+h)$ the sum of the series in (2), and of others similar to it with β, γ, &c. instead of a and different constants. Also, if we denote the second member of (3), without regard to the values of the constants, by $\phi(x, a)$, we should find that the form of $f(x)$, corresponding to the more general form of $f(x+h)$, is

$$\phi(x, a) + \phi(x, \beta) + \phi(x, \gamma), \text{ \&c.;}$$

which form includes all functions that are commonly called Algebraic. We shall always suppose the difference of any two of the indices β, γ, &c. to be fractions; for if it was an integer, the two series would unite into one.

8. The more general equation corresponding to (4) will be

$$f(x+h) = \left.\begin{aligned} &\frac{h^a}{\mathrm{P}(a)}\frac{d^a u}{dx^a} + \frac{h^{a-1}}{\mathrm{P}(a-1)}\frac{d^{a-1}u}{dx^{a-1}} + \ldots \\ &+ \frac{h^\beta}{\mathrm{P}(\beta)}\frac{d^\beta u}{dx^\beta} + \frac{h^{\beta-1}}{\mathrm{P}(\beta-1)}\frac{d^{\beta-1}u}{dx^{\beta-1}} + \ldots \\ &+ \frac{h^\gamma}{\mathrm{P}(\gamma)}\frac{d^\gamma u}{dx^\gamma} + \frac{h^\gamma}{\mathrm{P}(\gamma-1)}\frac{d^{\gamma-1}u}{dx^{\gamma-1}} + \ldots \end{aligned}\right\} \ldots(5),$$

&c.

with this provision, that whenever the index of differentiation is a positive integer, we reject from the differential coefficient terms involving fractional powers of x.

For since $\beta - a$ is a fraction, the index of x in every term of $\frac{d^a}{dx^a}\phi(x, \beta)$ is fractional, and therefore its coefficient finite, so that all these terms will disappear, together with the negative powers of x in $\frac{d^a}{dx^a}\phi(x, a)$, on being divided by $\mathrm{P}(a)$, which is infinite when a is not a positive integer. Hence

$$\frac{d^a u}{dx^a} = \frac{d^a}{dx^a}\phi(x, a),$$

and for the same reasons

$$\frac{d^{a-1}u}{dx^{a-1}} = \frac{d^{a-1}}{dx^{a-1}}\phi(x, a), \quad \frac{d^\beta u}{dx^\beta} = \frac{d^\beta}{dx^\beta}\phi(x, \beta), \text{ \&c.;}$$

so that, after expanding each of the functions

$$\phi(x+h, a), \quad \phi(x+h, \beta), \text{ \&c.}$$

we obtain equation (5).

9. As a function which requires the more general expansion (5) we may take $(x^{\frac{2}{3}} - a^{\frac{2}{3}})^{\frac{3}{2}}$. This expanded in descending powers of x by the binomial theorem is

$$x - \frac{3}{2} a^{\frac{2}{3}} x^{\frac{1}{3}} + \frac{3.1}{2.4} a^{\frac{4}{3}} x^{\frac{1}{3}} + \frac{3.1.1}{2.4.6} a^2 x^{-1} + \ldots\ldots$$

whence we see that the quantities α, β, γ, &c. are three in number, and that their values are 1, $\frac{1}{3}$, $-\frac{1}{3}$, for the rest of the indices in the last series differ from these by integers. The rest of the process is similar to that used in § 5 and § 6.

10. We have not yet obtained the most general expansion for $f(x+h)$. Suppose that $f(x) = \varepsilon^x \sqrt{x-a}$, and that we wish to expand $f(x+h)$ in a series that shall not fail when $x = a$. We may expand the factors ε^{x+h} and $\sqrt{x+h-a}$ separately, and multiply the series together. Now ε^{x+h} can only be expanded in ascending powers of h, and the other factor must be expanded in descending powers of h. By the multiplication of the two series, therefore, we should obtain one unlimited both ways, as long as x is not equal to a.

11. In order to obtain a general formula to suit such cases, assume for $f(x+h)$ a series unlimited both ways, namely,

$$\ldots\ldots + Bh^b + Ah^a + \mathrm{A}h^\alpha + \mathrm{B}h^\beta + \mathrm{C}h^\gamma + \ldots\ldots$$

where we suppose the indices to increase towards the right, and α to be the least positive index.

Equating the first differential coefficients with respect to x and h,

$$\ldots\ldots + \frac{dB}{dx} h^b + \frac{dA}{dx} h^a + \frac{d\mathrm{A}}{dx} h^\alpha + \frac{d\mathrm{B}}{dx} h^\beta + \ldots\ldots$$

$$= \ldots\ldots + aAh^{a-1} + \alpha\mathrm{A}h^{\alpha-1} + \beta\mathrm{B}h^{\beta-1} + \gamma\mathrm{C}h^{\gamma-1} + \ldots\ldots$$

If we suppose at first, for the sake of simplicity, that only one of the indices in the assumed series lies between 0 and 1, $\beta - 1$ must be the least positive index in the second member of the last equation, and by comparing the terms we obtain

$$b = a - 1, \quad a = \alpha - 1, \quad \alpha = \beta - 1, \quad \beta = \gamma - 1, \quad \&c.$$

$$\frac{dB}{dx} = aA, \quad \frac{dA}{dx} = \alpha\mathrm{A}, \quad \frac{d\mathrm{A}}{dx} = \beta\mathrm{B}, \quad \frac{d\mathrm{B}}{dx} = \gamma\mathrm{C}, \quad \&c.$$

and therefore, expressing the other quantities in terms of A and α,

$$f(x+h) = \ldots\ldots \alpha(\alpha-1) h^{\alpha-2} \int^2 \mathrm{A} dx^2 + \alpha h^{\alpha-1} \int \mathrm{A} dx + h^\alpha \mathrm{A} + \frac{h^{\alpha+1}}{\alpha+1} \frac{d\mathrm{A}}{dx} + \frac{h^{\alpha+2}}{(\alpha+1)(\alpha+2)} \frac{d^2\mathrm{A}}{dx^2} + \ldots\ldots (6).$$

12. It is easy to see what would be the consequence of supposing several of the indices in the assumed series to lie between 0 and 1, namely, that we should obtain for $f(x+h)$ the sum of several series similar to that in (6), but in which A would be

replaced by B, C, &c. successively, and a by β, γ, &c. *Such, then, is the most general expansion for* $f(x+h)$. It shews that $f(x+h)$, in its most general form, is the sum of several simpler functions of $x+h$, which may separately be expanded by the formula (6). It is enough to consider one of these functions at a time; we therefore return to that equation.

13. Let us examine the circumstances under which the series in (6) can become limited in either direction. The form of the series will not be altered if we suppose $h^{\alpha}\mathrm{A}$ to represent no longer the term with the least positive index, but the first or last term that does not vanish. Now all the terms to the left of that may be made to vanish by supposing $\alpha = 0$, but by no other supposition while the value of x is general. This leads us immediately to Taylor's series, which we thus see to be the only general expansion of $f(x+h)$ in ascending powers of h exclusively; and from there being no quantity in it left undetermined, such as a in (4), to be applicable to all functions.

The terms on the right of $h^{\alpha}\mathrm{A}$ may be made to vanish by supposing A a constant, and by that supposition only. The equation then becomes the same as (2).

14. It remains to determine the value of the quantity A in equation (6). This may be done in the following manner:

Let y be a function of x of the form (3), z any other function, and let y' and z' represent the values of these quantities when x becomes $x+h$. Then by (4)

$$y' = \frac{h^{\alpha}}{\mathrm{P}(a)}\frac{d^{\alpha}y}{dx^{\alpha}} + \frac{h^{\alpha-1}}{\mathrm{P}(a-1)}\frac{d^{\alpha-1}y}{dx^{\alpha-1}} + \frac{h^{\alpha-2}}{\mathrm{P}(a-2)}\frac{d^{\alpha-2}y}{dx^{\alpha-2}} + \ldots,$$

and, by Taylor's series,

$$z' = z + \frac{h}{1}\frac{dz}{dx} + \frac{h^2}{1.2}\frac{d^2z}{dx^2} + \ldots$$

The multiplication together of these two series will produce one unlimited in both directions, in which the coefficient of h^{α} will be

$$\frac{1}{\mathrm{P}(a)}\frac{d^{\alpha}y}{dx^{\alpha}}z + \frac{1}{1.\mathrm{P}(a-1)}\frac{d^{\alpha-1}y}{dx^{\alpha-1}}\frac{dz}{dx} + \frac{1}{1.2.\mathrm{P}(a-2)}\frac{d^{\alpha-2}y}{dx^{\alpha-2}}\frac{d^2z}{dx^2} + \ldots$$

$$= \frac{1}{\mathrm{P}(a)}\left(\frac{d^{\alpha}y}{dx^{\alpha}}z + \frac{a}{1}\frac{d^{\alpha-1}y}{dx^{\alpha-1}}\frac{dz}{dx} + \frac{a(a-1)}{1.2}\frac{d^{\alpha-2}y}{dx^{\alpha-2}}\frac{d^2z}{dx^2} + \ldots\right),$$

which, by Leibnitz's theorem generalized, is equal to

$$\frac{1}{\mathrm{P}(a)}\frac{d^{\alpha}.yz}{dx^{\alpha}}.$$

In the same manner it may be seen that the coefficient of $h^{\alpha-p}$ is

$$\frac{1}{\mathrm{P}(a-p)}\frac{d^{\alpha-p}yz}{dx^{\alpha-p}},$$

and that of h^{x+p},

$$\frac{1}{\mathrm{P}(a+p)}\,\frac{d^{x+p}yz}{dx^{x+p}},$$

if it be remembered that all the differential coefficients of y whose indices are greater than a are finite. The terms of the series, therefore, follow the same law as those of the series in (6); and if we suppose $yz = u$, the two series must be identical, so that by determining the quantity A, (6) becomes

$$f(x+h) = \ldots + \frac{h^{x-1}}{\mathrm{P}(a-1)}\,\frac{d^{x-1}u}{dx^{x-1}} + \frac{h^{x}}{\mathrm{P}(a)}\,\frac{d^{x}u}{dx^{x}} + \frac{h^{x+1}}{\mathrm{P}(a+1)}\,\frac{d^{x+1}u}{dx^{x+1}} + \ldots (7).$$

When it requires the sum of several series such as that in (6) to express the value of $f(x+h)$, the determination of A and the other corresponding quantities presents some difficulties, so that we must leave it to a future occasion, and perhaps to an abler analyst.

fff.

VI.—ON SOME ELEMENTARY PRINCIPLES IN THE APPLICATION OF ALGEBRA TO GEOMETRY.*

In pure Algebra, the independent existence of signs of affection is the immediate consequence of the universal applicability of its rules of operation. Thus, if the operation indicated in $a - b$ is always possible, whatever the relation between a and b may be, it follows that $a - (a + c)$, $a - a - c$, or (since $a - a = 0$) $-c$ must be a possible result of such an operation. The negative sign of affection, therefore, in Algebra, although it represents something not implied in the primary arithmetical definition of subtraction, is yet intimately connected with that operation, and can have no meaning distinct from that of subtraction in any case where subtraction becomes possible, or, in other words, $a + (-b)$ and $a - (+b)$ are identically equivalent.

But in applied Algebra the case is somewhat different. There may be quantities of specific natures, which by their reciprocal relations afford us an obvious interpretation of the signs of affection, and so enable us to define Addition and Subtraction as *operations* in a manner no less general than is done in Algebra itself; that is, we may lay down rules for the performing those operations, which rules shall not be limited, as in arithmetic, by the particular relations of the quantities, but shall be applicable in all cases. In such cases it is clear, that the independent sign of affection is the found-

* From a Correspondent.

ation, and not, as in Algebra, the consequence of the rule of operation; it exists in the very nature of the quantity which is to be subject to the rule, and is the condition which makes such a rule possible. In such cases, therefore, we shall have two distinct meanings of the negative sign $-$, as it indicates a quantity of a particular kind, or an operation to be performed; the only condition connecting the two being, that the adoption of either interpretation must lead to such results as may be considered *algebraical* equivalents. There may exist any specific diversity, as in the case of *geometrical* parallelism; but such diversity, being of a kind not comprehended in the algebraic symbols, forms no obstacle to their *algebraical* equivalence.

In the application of Algebra to Geometry, it is to be borne in mind, that the subject represented by the algebraical symbols is not geometrical extension, but rather extension combined with direction. It is not the distance *between* two points, but the distance of one point *rightwards* or *leftwards* of another. It is, in short, theoretical *motion*.

Quantities of the former, pure geometrical kind, have no opposite relation of plus or minus to their two extremities: subtraction applied to such quantities, is the expunging of a line, and when the magnitude has once been entirely expunged, the operation can be carried on no longer. The negative sign cannot be applied to distance alone, but to distance or progress in a given direction. A man who travels north, and afterwards travels the same or a greater distance south, has not traversed no space, nor a space less than zero, but his progress north is upon the whole either zero or negative. We say, therefore, that the quantities we have selected for the application of algebraical reasoning are essentially positive and negative, independently of any rules of addition and subtraction; they are quantities which are not adequately defined until we know whether they are preceded by $+$ or $-$, and we now proceed to inquire for necessary rules for the addition and subtraction of quantities of this description.

On Addition.

Let AB be a positive line drawn *from* A *towards* B, and represented by a or $+a$: required to add to it a positive line equal to $+b$.

Now all positive lines must be drawn in the same direction, say towards the right of the point of starting, or that point from which their description begins. To increase AB by a line drawn from left to right is impossible, except we start from the point B, or the *final* point in the motion by which AB was described. We must therefore *continue* the motion AB in the same direction, making $BC = b$, and AC, measured from A to C, and not from C to A, is the sum of the positive progressions $AB + BC$, or $a + b$.

If we wish to add a negative quantity, or $-b$, to AB, it is clear that the operation cannot be performed at A, the origin of AB; for a negative line drawn through A will increase the whole AB, and

if such a line were drawn equal to a in length, the result $a - a$, instead of being zero, would be a geometrical line equal twice AB in length; to *add* $-b$ to d, therefore, we must proceed in the same way as when we *add* $+b$ to a, *i.e.* we must draw BC from B in the negative direction, and as before the sum of the two is AC, a geometrical line, which is evidently positive or negative according as BC is $<$ or $>$ AC.

It appears, therefore, in both cases, that *addition* is an *operation* performed at B, the second extremity of AB.

On Subtraction.

From considerations such as the foregoing, we shall readily convince ourselves, that to subtract one positive line from another, or to perform the operation $a - (+b)$, can only be accomplished by drawing a line $AC = b$ from A, the origin of the minuend, and that the equivalent difference $AB - AC$ or $a - b$ is the distance CB *measured from the final point in the subtrahend to that of the minuend.*

This difference will obviously be negative when AC is $>$ AB, being always expressed by CB.

If AC be drawn from A negatively, then CB is

$$a - (-b) = a + b,$$

and is properly represented by a line equal to the sum of the two parts a and b, and positive in its description.

It may be remarked, that this is an independent proof of the rule for the concurrence of signs, and one whose place is supplied by no other. For that rule, as proved in Arithmetic, is confined to the concurrence of a double operation, and in Algebra it is an arbitrary assumption; but the proof we have here obtained is the only one for the case of the concurrence of the signs of operation with those of affection.

With respect to the expressions $a + (-b)$ and $a - (+b)$, it is evident that we obtain for them two geometrical lines, specifically different, but always identical in magnitude and direction; and as magnitude and direction are the only elements which are denoted by our algebraical system of representation, these two lines cannot be algebraically distinguished.

The rules here laid down for Addition and Subtraction, verify the algebraical results no less beautifully when the quantities involved are impossible symbols, and the direction of the lines rectangular.

If BC $\perp$ to AB be drawn upwards or downwards through B, we shall have AC as before

$$= a + (\pm b\sqrt{-1});$$

and if AC be similarly drawn through A we shall have

$$CB = a - (\pm b\sqrt{-1}).$$

So also, if AB be drawn in a direction inclined to the axis of a, and equal to $a + b\sqrt{-1}$, and if we draw through B, (and

therefore add) a line

$$BC = c + d\sqrt{-1};$$

then AC, the sum of these two, will readily be seen to represent

$$(a + c) + (b + d)\sqrt{-1};$$

and if AB and AC represent

$$a + b\sqrt{-1} \text{ and } c + d\sqrt{-1}$$

respectively, we get

$$CB = a - c + (b - d)\sqrt{-1}.$$

D.

VII.—NOTES ON THE UNDULATORY THEORY OF LIGHT.—No. II.

THE theorems of Fresnel respecting the motion of light in crystallized media, were demonstrated by him by means of two surfaces, *the surface of elasticity* and *the ellipsoid of construction*, which is related to the former by certain reciprocal relations. He was thus not only furnished with simple demonstrations of many important theorems, but was enabled to make use of the known properties of these surfaces for the purpose of suggesting corresponding properties of the rays and waves of light.

In the present state of the theory, it may be useful to show how these results may be obtained with as much or more simplicity from direct analysis; and without discarding the consideration of these surfaces, to use them rather to give clearness and connection to our results than as methods of demonstration.

Since the publication of Fresnel's Memoir in the 7th volume of the *Memoirs of the Institute*, the chief addition which has been made to his results is the beautiful discovery of conical refraction by Sir W. Hamilton: this will be shown to flow naturally from the equations we get in investigating the equation to the wave surface.

The method of proving the existence of the circle of plane contact on the wave surface is due to Mr. Greatheed: the expressions for l, m and n in terms of v_1, v_2, are given by Professor Sylvester in the *Philosophical Magazine*.

As in the preceding part, we shall take as coordinate axes the three axes of elasticity, and we shall assume that the forces excited by displacements equal to unity in those directions are a^2, b^2, c^2 respectively. We shall call a line or direction, the cosines of the inclinations of which to the axes are l, m, n, the *line* or *direction* $[l, m, n]$.

(1) Let then $lx+my+nz=0$ be the equation to the plane wave, and let (α, β, γ) be the direction of vibration; a displacement equal to unity will give rise to forces $a^2\alpha$, $b^2\beta$, $c^2\gamma$ parallel to the axes.

Resolve these forces along the displacement and perpendicular to it, and let the latter direction be determined by the cosines α', β', γ', so that (α, β, γ), $(\alpha', \beta', \gamma')$ being perpendicular to each other,

$$\alpha\alpha' + \beta\beta' + \gamma\gamma' = 0 \ldots\ldots\ldots\ldots (1),$$
$$\alpha'^2 + \beta'^2 + \gamma'^2 = 1 \ldots\ldots\ldots\ldots (2).$$

To determine α', β', γ', we have another relation arising from this, that the resultant, the displacement, and the line $(\alpha', \beta', \gamma')$ being all in the same plane, must all be perpendicular to some line (f, g, h), or

$$\left.\begin{aligned} f\alpha + g\beta + h\gamma &= 0 \\ f\alpha' + g\beta' + h\gamma' &= 0 \\ fa^2\alpha + gb^2\beta + hc^2\gamma &= 0 \end{aligned}\right\} \ldots\ldots (A).$$

Eliminating f, g, h by cross multiplication,

$$\frac{\alpha'}{\alpha}(b^2 - c^2) + \frac{\beta'}{\beta}(c^2 - a^2) + \frac{\gamma'}{\gamma}(a^2 - b^2) = 0 \ldots\ldots (3).$$

If $(\alpha', \beta', \gamma')$ be perpendicular to the plane,

$$\alpha' = l, \quad \beta' = m, \quad \gamma' = n,$$

and making these substitutions, the equations (1), (2), (3) will determine α, β, γ.

The part of the resultant resolved along the displacement is

$$a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2,$$

and, as the other part, which is perpendicular to the direction of displacement, is also perpendicular to the front of the wave, it may, as we have shown before, be neglected, we have a wave proceeding with a velocity

$$v = \sqrt{a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2} \ldots\ldots\ldots (4).$$

(2) If we multiply together the equations (1) and (3), put under the form

$$\left.\begin{aligned} l\alpha + m\beta &= -n\gamma \\ \frac{l}{\alpha}(b^2 - c^2) + \frac{m}{\beta}(c^2 - a^2) &= -\frac{n}{\gamma}(a^2 - b^2) \end{aligned}\right\} \ldots\ldots (B),$$

we get the equation

$$lm\left\{\frac{\beta}{\alpha}(b^2 - c^2) + \frac{\alpha}{\beta}(c^2 - a^2)\right\} + \&c. = 0 \ldots\ldots (5),$$

a quadratic in $\frac{\alpha}{\beta}$, the last term of which, $\frac{b^2 - c^2}{c^2 - a^2}$, is negative, and therefore the roots are real; the same is the case in the equation we should find for $\frac{\gamma}{\beta}$: and thus the above equations give two real directions (α, β, γ).

Let a_1, a_2; β_1, β_2; γ_1, γ_2, be the roots

$$\frac{a_1 a_2}{\beta_1 \beta_2} = \frac{b^2 - c^2}{c^2 - a^2},$$

$$\text{or } \frac{a_1 a_2}{b^2 - c^2} = \frac{\beta_1 \beta_2}{c^2 - a^2} = \frac{\gamma_1 \gamma_2}{a^2 - b^2};$$

$$\text{hence } a_1 a_2 + \beta_1 \beta_2 + \gamma_1 \gamma_2 = 0 \ldots\ldots\ldots(6),$$

and the two directions are perpendicular to each other.

Whatever then be the nature of the vibration of the particles in the plane wave, we may resolve it in these two directions, and we shall have two waves, whose vibrations are perpendicular to each other, and proceed with different velocities v_1 and v_2, determined from equation (4): that is to say, *any plane disturbance will produce two plane waves polarized in planes at right angles to each other, and of different refrangibilities.*

(3) v_1, v_2 are the roots of a quadratic equation, which may easily be found from the equations (A) by substituting l, m, n for α', β', γ'. The method is the same as that in page 7 of our last Number, and the result is the same, namely,

$$\frac{l^2}{a^2 - v^2} + \frac{m^2}{b^2 - v^2} + \frac{n^2}{c^2 - v^2} = 0 \ldots\ldots\ldots(7).$$

(4) Having determined the velocities of propagation of a plane disturbance, the form of the wave surface, or that surface into which a disturbance at any point will diverge, may be found.

If we suppose the disturbance to be caused by an infinite number of plane waves passing through the origin, each too faint to cause any impression on the senses of itself; then each of these waves will be propagated with the velocities we have just determined, and the form of the sensible disturbance at any instant will be the locus of their ultimate intersections, or the surface touched by all the plane waves.

At the end of a unit of time the equation to one of these planes is

$$lx + my + nz = v$$

with the relations

$$\left.\begin{aligned} l^2 + m^2 + n^2 &= 1 \\ \frac{l^2}{a^2 - v^2} + \frac{m^2}{b^2 - v^2} + \frac{n^2}{c^2 - v^2} &= 0 \end{aligned}\right\} \ldots (C).$$

The process of elimination will be found in the *Cambridge Transactions*, Vol. vi. Part 1.

(5) The following three equations are found after eliminating the differentials,

$$\left.\begin{aligned} x &= \frac{lv}{a^2 - v^2}(a^2 - r^2) \\ y &= \frac{mv}{b^2 - v^2}(b^2 - r^2) \\ z &= \frac{nv}{c^2 - v^2}(c^2 - r^2) \end{aligned}\right\} \text{.....(D),}$$

in which $r^2 = x^2 + y^2 + z^2$,

and the resulting equation to the wave surface is

$$\frac{a^2x^2}{a^2 - r^2} + \frac{b^2y^2}{b^2 - r^2} + \frac{c^2z^2}{c^2 - r^2} = 0 \text{ .. ...(8).}$$

If X, Y, Z be the angles which a radius vector makes with the coordinate axes, the polar equation is

$$\frac{a^2\cos^2 X}{a^2 - r^2} + \frac{b^2\cos^2 Y}{b^2 - r^2} + \frac{c^2\cos^2 Z}{c^2 - r^2} = 0\text{.....(9).}$$

6. We shall now discuss some of the equations we have obtained.

The two equations (B) will in general determine two of the quantities $\frac{a}{\beta}, \frac{\gamma}{\beta}$, and the directions of the vibration lines will be definite; but if one of the quantities l, m, n, as for instance m, is zero, the two equations will become identical, provided

$$n^2(a^2 - b^2) - l^2(b^2 - c^2) = 0 \text{ .....(10),}$$

and thus the direction of the lines of vibration become indeterminate, if

$$l = \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}, \quad m = 0, \quad n = \sqrt{\frac{b^2 - c^2}{a^2 - c^2}} \text{ ... (E).}$$

These values of l and n are real, provided a, b, c are in the order of magnitude, but not otherwise; and thus we should not get a similar property by making $l = 0$ or $n = 0$.

Putting $a^2 - b^2 + b^2 - v^2$ and $b^2 - v^2 - (b^2 - c^2)$ for $a^2 - v^2$ and $c^2 - v^2$ in equation (7), we get

$$(b^2 - v^2)^2 + (b^2 - v^2) \times$$
$$[n^2(a^2 - b^2) - l^2(b^2 - c^2) + m^2\{a^2 - b^2 - (b^2 - c^2)\}]$$
$$- m^2(a^2 - b^2)(b^2 - c^2) = 0 \text{ ..... (11),}$$

which on our supposition reduces to

$$(b^2 - v^2)^2 = 0.$$

Thus the two values of v^2 become equal to each other and to b^2. There is no separation of the plane wave; and therefore for the two directions determined by equations (E), and for these only, a single plane wave may be propagated. The normal to this plane is then a direction or *line of single normal velocity*.

7. If equation (9) be put under the form

$$\frac{\cos^2 X}{\frac{1}{a^2}-\frac{1}{r^2}}+\frac{\cos^2 Y}{\frac{1}{b^2}-\frac{1}{r^2}}+\frac{\cos^2 Z}{\frac{1}{c^2}-\frac{1}{r^2}}=0,$$

it becomes exactly similar in form to equation (7), and we at once infer that the two values of r^2 become equal to each other and to b^2, provided

$$\cos X=\sqrt{\frac{\frac{1}{a^2}-\frac{1}{b^2}}{\frac{1}{a^2}-\frac{1}{c^2}}},\quad \cos Y=0,\quad \cos Z=\sqrt{\frac{\frac{1}{b^2}-\frac{1}{c^2}}{\frac{1}{a^2}-\frac{1}{c^2}}},$$

$$\text{or } \cos X=\frac{c}{b}\sqrt{\frac{a^2-b^2}{a^2-c^2}},\quad \cos Y=0,\quad \cos Z=\frac{a}{b}\sqrt{\frac{b^2-c^2}{a^2-c^2}}\ldots(F),$$

and the radius vector, which in general cuts the surface in two points on the same side of the origin, will in this direction cut it only in one. This then is a direction or *line of single ray velocity.*

8. These two directions, to which we have given Sir William Hamilton's names, *the lines of single normal velocity*, and *the lines of single ray velocity*, are nearly coincident in most crystals. They were not distinguished by observation before Fresnel's theory, but, from some remarkable optical properties which they possess, they received the common name of *optic axes.*

When a, b and c are different, the equations (E) and (F) give two lines inclined at equal angles to the plane of xy, and crystals of this kind were called *biaxal;* if $b=a$ or $=c$, the two directions coincide, and such crystals were called *uniaxal.* Fresnel has given the name of optic axes sometimes to one set and sometimes to the other, and considerable confusion has arisen from the ambiguity. We shall use the terms *wave axis* and *ray axis;* names which prevent ambiguity, and which appear sufficiently to explain their own meaning.

9. By comparing equations (7) and (E) with equations (9) and (F), we see that, supposing the direction of a ray to be the same as that of a wave normal, the equations for determining the wave velocity v and the directions of the wave axes, are changed into those for determining the ray velocity r and the directions of the ray axes, by substituting $\frac{1}{r}$ for v and for a, b, c, their reciprocals. And hence, that every relation between the wave velocity and the directions of the wave normal and wave axes, will give a corresponding relation for the ray.

If we have

$$\phi\,(l,\ m,\ n,\ v,\ a,\ b,\ c,\ \text{direction of wave axis})=0,$$

we derive at once the corresponding relation

$$\phi\,(\cos X,\ \cos Y,\ \cos Z,\ \frac{1}{r},\ \frac{1}{a},\ \frac{1}{b},\ \frac{1}{c},\ \text{direction of ray axis}) = 0 \ldots\ldots\ldots\ldots\ldots (G).$$

10. Equation (11) gives us the inclination of the wave normal to the axes in terms of the two wave velocities v_1, v_2; for as the last term is the product of the two roots of the equation

$$(b^2 - v_1^2)\,(b^2 - v_2^2) = -\,m^2\,(a^2 - b^2)\,(b^2 - c^2),$$

$$\text{or } m = \sqrt{\frac{(b^2 - v_1^2)\,(v_2^2 - b^2)}{(a^2 - b^2)\,(b^2 - c^2)}};$$

similarly,

$$l = \sqrt{\frac{(a^2 - v_1^2)\,(a^2 - v_2^2)}{(a^2 - b^2)\,(a^2 - c^2)}},\quad n = \sqrt{\frac{(v_1^2 - c^2)\,(v_2^2 - c^2)}{(a^2 - c^2)\,(b^2 - c^2)}} \ldots (H).$$

From these expressions we may find ι, ι', the angles between the wave normal and the wave axes, in terms of the velocities v_1, v_2,

$$\cos\iota = l\sqrt{\frac{a^2 - b^2}{a^2 - c^2}} + n\sqrt{\frac{b^2 - c^2}{a^2 - c^2}}.$$

Substituting the above values of l and n, this becomes

$$(a^2 - c^2)\cos\iota = \sqrt{(a^2 - v_1^2)\,(a^2 - v_2^2)} + \sqrt{(v_1^2 - c^2)\,(v_2^2 - c^2)};$$

squaring, putting $a^2 - c^2 = (v_1^2 - c^2)$ for $a^2 - v_1^2$, and making a similar substitution for $v_1^2 - c^2$, the equation reduces to

$$(a^2 - c^2)^2\cos^2\iota = (a^2 - c^2)^2 - \{\sqrt{(a^2 - v_1^2)\,(v_2^2 - c^2)} - \sqrt{(a^2 - v_2^2)\,(v_1^2 - c^2)}\}^2,$$

$$(a^2 - c^2)\sin\iota = \sqrt{(a^2 - v_1^2)\,(v_2^2 - c^2)} - \sqrt{(a^2 - v_2^2)\,(v_1^2 - c^2)}.$$

The expression for $\sin\iota'$ will only differ by the sign of one of the radicals; the product of the two expressions will therefore be the difference of the squares of the radicals, or

$$(a^2 - c^2)^2\sin\iota\sin\iota' = \pm\,(a^2 - c^2)\,(v_2^2 - v_1^2),$$

$$\text{or } v_2^2 - v_1^2 = \pm\,(a^2 - c^2)\sin\iota\sin\iota' \ldots\ldots (12).$$

If ε, ε' be the angles between the ray and the ray axis, we deduce from the above at once

$$\frac{1}{r_2^2} - \frac{1}{r_1^2} = \pm\left(\frac{1}{a^2} - \frac{1}{c^2}\right)\sin\varepsilon\sin\varepsilon' \ldots\ldots\ldots (13),$$

or the difference of the reciprocals of the squares of the ray velocities is proportional to the product of the series of the inclinations of the ray to the ray axes. This law was discovered experimentally by Biot. In Airy's tract on the *Undulatory Theory*, No. 123, this property is inadvertently stated as belonging to the inclination of the ray to the *wave axes*.

11. We shall now proceed to consider the form of the wave surface given by equation (8).

The equations to its intersections with the coordinate planes will consist, in each case, of a circle and ellipse. In the plane of xy the circle is contained within the ellipse; in the plane of yz the ellipse is contained within the circle; in the plane of xz the ellipse and circle cut each other, being given by the equation

$$(r^2 - b^2)\left(\frac{x^2}{c^2} + \frac{z^2}{a^2} - 1\right) = 0.$$

The point of intersection of the two curves is the extremity of the ray axis. The point where a common tangent to the two curves touches the circle is the extremity of the wave axis.

Fig. (2) is a section of the surface by the plane of xz, D is the extremity of the ray axis, and F, the point where the common tangent EF touches the circle, the extremity of the wave axis.

12. The equations (D) determine the point at which a tangent plane of given position touches the wave surface; or if the point of contact be given, they determine the position of the tangent plane: that is, if l, m, n and v be given, they determine x, y, z; conversely, if x, y, z be given, the equations (D) with

$$lx + my + nz = v,$$

will determine l, m, n, v.

If for any values of l, m, n, v, one of these equations is satisfied independently of the values of x, y, and z, these equations will determine, not a point, but a curve of contact with the tangent plane. Also, if for any values of x, y, z, one of the equations is satisfied independently of l, m, n, v, we shall get only two relations between l, m, n, and there will be an infinite number of tangent planes at that point, giving a cone of contact.

13. If $m = 0$, $v^2 = b^2$, the second equation is satisfied independently of x, y, z: this is the case we have before considered of the plane of single normal velocity.

The equations (D) are the equations of three spheres; hence, any two of them determine a circle: and thus it appears, that that plane wave which is propagated without separation, or along the wave axis, touches the wave surface in a circle, the position and magnitude of which we proceed to find.

We have, as before,

$$l = \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}, \quad n = \sqrt{\frac{b^2 - c^2}{a^2 - c^2}};$$

substituting these values, the two equations (D) become

$$x = \frac{b}{\sqrt{(a^2 - b^2)(a^2 - c^2)}}(a^2 - r^2),$$

$$z = \frac{b}{\sqrt{(a^2 - c^2)(b^2 - c^2)}}(r^2 - c^2);$$

or putting $x^2 + y^2 + z^2$ for r^2,

$$x^2 + y^2 + z^2 + \frac{x}{b}\sqrt{(a^2 - b^2)(a^2 - c^2)} - a^2 = 0,$$

$$x^2 + y^2 + z^2 - \frac{z}{b}\sqrt{(a^2 - c^2)(b^2 - c^2)} - c^2 = 0.$$

If we eliminate z, we get for the projection on the plane of xy

$$\frac{a^2-c^2}{b^2-c^2}\left(x - \sqrt{\frac{a^2-b^2}{a^2-c^2}}\,\frac{b^2+c^2}{2b}\right)^2 + y^2 = \frac{(a^2-b^2)(b^2-c^2)}{4b^2},$$

the equation to an ellipse, of which the axis parallel to y is the radius of the circle of contact,

$$= \frac{\sqrt{(a^2 - b^2)(b^2 - c^2)}}{2b} \quad \ldots\ldots\ldots\ldots (14).$$

This equation also gives one coordinate of the centre of the circle,

$$x = \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}\,\frac{b^2 + c^2}{2b} \quad \ldots\ldots\ldots (15).$$

Similarly we should find

$$z = \sqrt{\frac{b^2 - c^2}{a^2 - c^2}}\,\frac{a^2 + b^2}{2b} \quad \ldots\ldots\ldots (16).$$

And thus the position and magnitude of the circle of contact is completely determined.

From the double sign $\pm$ which these radicals involve, we see that there will be four such circles on the wave surface, one at each end of the two wave axes.

14. If $y = 0$, $r^2 = b^2$, the second equation is satisfied independently of l, m, n: this is the case at the extremity of the ray axis, where

$$x = c\sqrt{\frac{a^2 - b^2}{a^2 - c^2}}, \quad z = a\sqrt{\frac{b^2 - c^2}{a^2 - c^2}}.$$

Substituting these values, the first of equations (D) becomes

$$c\sqrt{\frac{a^2 - b^2}{a^2 - c^2}} = \frac{lv}{a^2 - v^2}(a^2 - b^2),$$

$$\text{or } a^2 - v^2 = \frac{lv}{c}\sqrt{(a^2 - b^2)(a^2 - c^2)}:$$

$$\text{similarly, } v^2 - c^2 = \frac{nv}{a}\sqrt{(a^2 - c^2)(b^2 - c^2)};$$

and eliminating v, we get the equation

$$l^2(b^2 - c^2) + m^2(a^2 - c^2) + n^2(a^2 - b^2)$$

$$- ln\sqrt{(a^2 - b^2)(b^2 - c^2)}\,\frac{a^2 + c^2}{ac} = 0 \ldots\ldots (17),$$

which, with the equation $l^2 + m^2 + n^2 = 1$, gives the relations between l, m, n.

At this point there are an infinite number of tangent planes, whose ultimate intersections give a tangent cone. The equation found above, where x, y, z are put for l, m, n, is the equation to the surface described by a line always normal to these planes.

15. To find the equation to the tangent cone, call the above equation $u = 0$, $lx + my + nz = 0$ is a plane through the origin parallel to a tangent plane.

We have, therefore,

$$xdl + ydm + zdn = 0,$$

$$ldl + mdm + ndn = 0,$$

$$\frac{du}{dl}\,dl + \frac{du}{dm}\,dm + \frac{du}{dn}\,du = 0;$$

$$\text{therefore } \frac{du}{dl} + \lambda l + \mu x = 0,$$

$$\frac{du}{dm} + \lambda m + \mu y = 0,$$

$$\frac{du}{dn} + \lambda n + \mu z = 0.$$

Multiplying by l, m, n, and adding, $\lambda = 0$. We have then to eliminate μ, l, m, n between

$$u = 0,\quad \frac{du}{dl} + \mu x = 0,\quad \frac{du}{dm} + \mu y = 0,\quad \frac{du}{dn} + \mu z = 0:$$

this is easily done, and the resulting equation

$$\frac{x^2}{b^2 - c^2} - \frac{(a^2 - c^2)\,y^2}{4a^2c^2} + \frac{z^2}{a^2 - b^2}$$

$$+ \frac{xz}{\sqrt{(a^2 - b^2)\,(b^2 - c^2)}}\,\frac{a^2 + c^2}{ac} = 0 \ldots\ldots\ldots (18)$$

is the equation to the tangent cone, the origin being at its vertex.

This is an elliptic cone; the axes of any section made by a plane perpendicular to its axis, may be found by transforming the co-ordinates so that the term involving xz may vanish.

The angle of the cone in the plane of xz may be found from either of the equations (17) or (18). If we make $m = 0$ in (17), and solve for $\frac{l}{n}$, we find

$$\frac{l}{n} = \sqrt{\frac{a^2 - b^2}{b^2 - c^2}}\,\frac{a^2 + c^2 \pm (a^2 - c^2)}{2ac},$$

and the tangent of the angle sought

$$= \frac{\text{difference of roots}}{1 + \text{product of roots}}$$

$$= -\frac{\sqrt{\frac{a^2 - b^2}{b^2 - c^2}} \cdot \frac{a^2 - c^2}{ac}}{1 + \frac{a^2 - b^2}{b^2 - c^2}}$$

$$= -\frac{\sqrt{(a^2 - b^2)(b^2 - c^2)}}{ac}.$$

At the extremity of the ray axis there is a *cusp*, or rather a *nodal point*, somewhat resembling the vertex of a very obtuse double cone. Perhaps a clearer conception of this surface may be formed by supposing an ellipse, as in fig. (3), to revolve round a diameter AB, which is not one of the axes. At A and B there will be two nodal points, each having an infinite number of tangent planes forming two cones of contact; and the two tangent planes FF′, GG′ will touch the surface in two circles. At the points D and E there is a cusp or node of a different kind: here there will be only two different tangent planes, and the tangent planes HH′, KK′ will touch the surface in two points. The form of the surface at A and B will represent the form of the node of the wave surface very nearly, but not exactly, unless the ellipse be supposed to change its form while revolving.

From the existence of the four tangent cones at the extremities of the ray axes, and the four circles of plane contact corresponding to the wave of single normal velocity, Sir William Hamilton draws the following inferences, which have been fully confirmed by the experiments of Professor Lloyd.

1. That if a ray be refracted into the crystal so as to proceed along the ray axis, then, as the position of the tangent plane at the extremity of the axis is indeterminate, it will at emergence be refracted into an infinite number forming a cone, and giving *external conical refraction.*

2. That if a single ray be incident externally at an angle corresponding to the wave axis, it will be refracted internally into an infinite number forming a cone, and giving *internal conical refraction.* If the faces of the crystal be parallel, the rays will emerge parallel to the incident ray, forming an external cylinder of rays.

A. S.

VIII.—SOLUTIONS OF SOME PROBLEMS IN TRANSVERSALS.

THE name of Transversals was given by Carnot to lines considered in their relations of mutual intersection. Many of their properties are very curious, and form interesting problems in Analytical Geometry, though it was not in this way that Carnot considered them. His method was, to proceed step by step from the more simple properties to the more complicated; but it seems better to consider each independently.

1. The following problem, under a slightly different form, was given in one of the Problem Papers for 1836.

If two lines AB, CD intersect in O so that AB is bisected, and if the lines AC, BD meet, when produced, in E, and AD, BC in F, then the line EF is parallel to AB.

Take O as the origin, and AB, CD as the axes of x and y.

Let $OB = a$, $OA = -a$, $OD = b$, $OC = -c$.

The equations to BD and AC are

$$\frac{x}{a} + \frac{y}{b} = 1, \quad \frac{x}{a} + \frac{y}{c} = -1.$$

At their intersection the equations may be combined in any manner; therefore, subtracting them,

$$y\left(\frac{1}{b} - \frac{1}{c}\right) = 2 \ldots\ldots\ldots\ldots (1).$$

The equations to BC and AD are

$$\frac{x}{a} - \frac{y}{c} = 1, \quad \frac{x}{a} - \frac{y}{b} = -1.$$

At their intersection, subtracting them,

$$y\left(\frac{1}{b} - \frac{1}{c}\right) = 2 \ldots\ldots\ldots\ldots (2),$$

which is the same as the equation (1), and therefore is the equation to the line passing through the points of intersection, that is, to EF; and from its form it is evident that it is parallel to the axis of x, that is, to AB.

2. If three lines be drawn from the angles of a triangle A, B, C, (fig. 5), through one point O, and meeting the sides of the triangle in P, Q, R, the sides of the triangle P, Q, R will, when produced, meet those of A, B, C in three points, which are in the same straight line.

Take L, the point of intersection of AB and PQ, as the origin, and these lines as the axes of x and y.

Let $LB = a_1$, $LR = a_2$, $LA = a_3$, $LP = b_1$, $LQ = b_2$.

The equations to BC and QR are

$$\frac{x}{a_1} + \frac{y}{b_1} = 1, \quad \frac{x}{a_2} + \frac{y}{b_2} = 1.$$

At their intersection we have, by subtraction,

$$x\left(\frac{1}{a_1} - \frac{1}{a_2}\right) + y\left(\frac{1}{b_1} - \frac{1}{b_2}\right) = 0 \ldots\ldots (1).$$

Again, the equations to AC and PR are

$$\frac{x}{a_3} + \frac{y}{b_2} = 1, \quad \frac{x}{a_2} + \frac{y}{b_1} = 1;$$

and at their intersection we have, by subtraction,

$$x\left(\frac{1}{a_2} - \frac{1}{a_3}\right) + y\left(\frac{1}{b_1} - \frac{1}{b_2}\right) = 0 \ldots\ldots (2).$$

Again, combining by addition the equations to BC and AC, we get at their point of intersection C,

$$x\left(\frac{1}{a_1} + \frac{1}{a_3}\right) + y\left(\frac{1}{b_1} + \frac{1}{b_2}\right) = 2 \ldots\ldots (3).$$

Also, the equations to AP and BQ are

$$\frac{x}{a_3} + \frac{y}{b_1} = 1, \quad \frac{x}{a_1} + \frac{y}{b_2} = 1;$$

and at their intersection we have, by addition,

$$x\left(\frac{1}{a_1} + \frac{1}{a_3}\right) + y\left(\frac{1}{b_1} + \frac{1}{b_2}\right) = 2 \ldots\ldots (4),$$

which is the same as (3), and therefore is the equation to COR, which passes through the two points of intersection.

If in this equation we make $y = 0$, we have $x = \frac{2a_1a_3}{a_1 + a_3}$, but in this case $x = LR = a_2$, therefore

$$a_2 = \frac{2a_1a_3}{a_1 + a_3}, \quad \text{or} \quad \frac{1}{a_1} - \frac{1}{a_2} = \frac{1}{a_2} - \frac{1}{a_3};$$

therefore equations (1) and (2) coincide, and represent the line passing through M and N, and as from the form of the equation it evidently passes through the origin, the three points L, M, N are in one straight line.

In somewhat a similar manner it might be shown, that if three circles be touched, two and two, by pairs of tangents, the points of intersection of these tangents are in one straight line. But we shall pass on to another problem.

3. If from any point A (fig. 6) in a line of indefinite length equal distances AQ_1, Q_1Q_2, &c. be measured off, and other equal distances AP_1, P_1P_2, &c. in the opposite direction; and if Q_1, Q_2, Q_3, &c. be joined by straight lines with a point O_1,

and P_1, P_2, P_3, &c. with a point O_2, O_1 and O_2 being situate in a line parallel to the given line; then the corresponding points of intersection of these lines will lie in straight lines forming two bundles of lines converging to two points O′, O″, situate in O_1O_2.

Take A as the origin, AO_1 as the axis of x, AQ as the axis of y.

Let $AO_1 = m$, $O_1O_2 = c$, $AQ = a$, $AP = b$.

Let one of the lines O_2P_n meet the axis of x in a point a_n; then by similar triangles it is easy to show that

$$\frac{1}{A_n} = \frac{1}{m} + \frac{c}{mnb}.$$

The equation to O_1Q_r is

$$\frac{x}{m} + \frac{y}{ra} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and the equation to O_2P_s is

$$\frac{x}{m} + \frac{cx}{msb} - \frac{y}{sa} = 1 \quad \ldots\ldots\ldots\ldots (2).$$

Combining them by subtraction, after multiplying by r and s,

$$(s - r)\frac{x}{m} + \frac{cx}{mb} - y\left(\frac{1}{a} + \frac{1}{b}\right) = s - r \quad \ldots\ldots\ldots (3),$$

which gives a relation between the coordinates of the point of intersection of the lines. We shall arrive at the same result for all pairs of lines for which $s - r$ is the same; consequently equation (3) will represent a line passing through the point of intersection of each pair. Now, let $x = m$, then

$$y = \frac{ac}{a + b},$$

which, being independent of r and s, will be the same for all the lines represented by (3) for which $s - r$ is different, and therefore they all converge to one point, which it is easy to see lies between O_1 and O_2, as the value of y is less than that of c. If we make $x = 0$ in (3), we get

$$y = (s - r)\left(\frac{1}{a} + \frac{1}{b}\right);$$

and as $s - r$ increases by unity, the distance between the points where the different lines diverging from O′ cut the axis of y is half the harmonic mean between a and b.

If, instead of subtracting, we add equations (1) and (2), we get

$$(r + s)\frac{x}{m} + \frac{cx}{mb} + y\left(\frac{1}{a} - \frac{1}{b}\right) = r + s,$$

which, as before, may be shown to be the equation to a line passing through the intersections of all lines for which $r + s$ is the same.

And if $r+s$ vary, we find, as before, that the different lines converge to a point O'', such that

$$O_1O'' = -\frac{ac}{b-a}.$$

If $b > a$, O'' is situate in O_2O_1 produced. If $a = b$, O_1O'' becomes infinite, and the lines are then parallel to AQ. If $b < a$, the point O'' is above O_2. The distance between the points in which the lines diverging from O'' cut the axis of y is $\frac{1}{a} - \frac{1}{b}$. It is easy also to show, that all the lines measuring from the points of convergence are divided harmonically: the equation to any line, as $O_2Q_{,}$, is

$$\frac{x}{m} + \frac{cx}{msb} - \frac{y}{sa} = 1,$$

$$\text{when } y = 0, \quad \frac{x}{m}\left(1 + \frac{c}{sb}\right) = 1;$$

$$\text{whence } \frac{1}{O_1s} = \frac{1}{m-x} = \frac{1}{m} + \frac{sb}{mc};$$

$$\text{similarly, } \frac{1}{O_1r} = \frac{1}{m} + \frac{rb}{mc},$$

and so on for the others; and as these are in arithmetic progression, O_1s, O_1r, &c. are in harmonic progression. Instead of taking O_1A as axis, we might have taken any other line, as $O_1Q_{,}$, and the result would be the same. And in like manner we might proceed, taking O_2 as the origin, as also O', O'', and, as the cases are similar, we should get the same result. Hence the property holds for all the lines in the figure. For other problems in Transversals the reader is referred to Carnot's Memoir, or to a small work by Brianchon on the same subject, where he will find some practical applications of the theory.

F.

IX.—ON A NEW FORM OF EQUILIBRIUM OF A REVOLVING FLUID.

In a letter addressed to the French Institute in 1834, M. Jacobi announced the curious discovery, that the form of equilibrium of a fluid mass, consisting of particles attracting each other according to the law of the inverse square of the distance, and revolving about an axis, may be an ellipsoid with three different axes, the shortest being the axis of revolution. A demonstration of this theorem is

given by M. Liouville in the *Journal de l' Ecole Polytechnique*, vol. xiv.; but as this work is not easy to be procured, we think a notice of this remarkable discovery will be acceptable to many of our readers. The following demonstration differs considerably from that of Liouville.

Let the fluid mass revolve about the axis of z with an angular velocity ω, and let X, Y, Z be the resolved attractions on a particle at (x, y, z).

The equation to the free surface of the fluid is

$$(X - \omega^2 x)\,dx + (Y - \omega^2 y)\,dy + Z dz = 0.$$

As we know that in the case of an ellipsoid X is of the form Ax, $Y = By$, $Z = Cz$, where A, B, C are independent of the coordinates of the point attracted, we see that this is in the form of the differential equation to an ellipsoid whose axes are a, b, c, or

$$\frac{x dx}{a^2} + \frac{y dy}{b^2} + \frac{z dz}{c^2} = 0,$$

provided

$$A - \omega^2 = \frac{\lambda}{a^2}, \quad B - \omega^2 = \frac{\lambda}{b^2}, \quad C = \frac{\lambda}{c^2}.$$

It remains to find whether these relations will give real values for the axes and angular velocity when a and b are different.

$$\text{Now } A = \frac{3M}{a}\int_0^{\infty} \frac{s^2 ds}{\sqrt{\{a^2 + (b^2 - a^2)s^2\}\{a^2 + (c^2 - a^2)s^2\}}}.$$

Transform this definite integral into one whose limits are 0 and $\frac{1}{0}$, by putting $s = \frac{a}{\sqrt{a^2 + u}}$, and we get

$$A = \frac{3}{2} M \int_0^{\infty} \frac{du}{(a^2 + u)\sqrt{(a^2 + u)(b^2 + u)(c^2 + u)}};$$

calling the quantity under the radical D,

$$A = \frac{3}{2} M \int_0^{\infty} \frac{du}{(a^2 + u)D},$$

$$\text{similarly } B = \frac{3}{2} M \int_0^{\infty} \frac{du}{(b^2 + u)D}, \quad C = \frac{3}{2} M \int_0^{\infty} \frac{du}{(c^2 + u)D}.$$

Eliminating ω^2 and λ, we get the relation

$$a^2 b^2 (B - A) - (a^2 - b^2) c^2 C = 0,$$

$$\text{or } (a^2 - b^2) \int_0^{\infty} \frac{du}{D} \left\{ \frac{1}{\left(1 + \frac{u}{a^2}\right)\left(1 + \frac{u}{b^2}\right)} - \frac{1}{\left(1 + \frac{u}{c^2}\right)^2} \right\} = 0.$$

If a be different from b, the relation between the axes must satisfy the equation

$$\int_0^\infty \frac{udu}{\mathrm{D}^3}\left(\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{c^2}+\frac{u}{a^2b^2}\right)=0.$$

Supposing a and b given, this may be considered as an equation to determine c, and as it is negative when $c=0$, and positive when $c=\frac{1}{0}$, there will be at least one real root of c.

As the above integral is zero, and $\frac{u}{\mathrm{D}^3}$ is always positive, and

$$\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{c^2}+\frac{u}{a^2b^2}$$

is positive if u be large enough; when u is small, this factor must be negative, or

$$\frac{1}{c^2}>\frac{1}{a^2}+\frac{1}{b^2},$$

and therefore c^2 is less than a^2 or b^2.

To determine the angular velocity, we have

$$(a^2-b^2)\,\omega^2 = \mathrm{A}a^2-\mathrm{B}b^2$$

$$=\frac{3}{2}\mathrm{M}\,(a^2-b^2)\int_0^\infty \frac{udu}{(a^2+u)\,(b^2+u)\,\mathrm{D}};$$

if a be different from b,

$$\omega^2=\frac{3}{2}\mathrm{M}\int_0^\infty \frac{udu}{(a^2+u)\,(b^2+u)\,\mathrm{D}},$$

a positive quantity. Hence ω is a possible quantity, and there being at least one possible value of c, the problem is possible.

H. M.

X.—RADIUS OF ABSOLUTE CURVATURE.

THE following is a simple method of finding the expression for the radius of absolute curvature, taking the arc as the independent variable. Let PQ, QR (Fig. 4) be two consecutive elements of the curve, and let us suppose them to be equal; this is the same thing as making ds constant, or s the independent variable. If we complete the parallelogram PQRS, the centre of curvature will lie in QS produced, which bisects the angle PQR, and if O be the centre

of curvature it will be the centre of a circle passing through P, Q, R, so that putting $QO = \rho$ and $QS = \lambda$,

$$QR^2 = ds^2 = \lambda\rho.$$

Let α, β, γ be the angles QO makes with the axes, then projecting QS on the axis of x, we have

$$QS \cos \alpha = RT \cos \alpha = M'N'.$$

Now $M'N' = NN' - NM' = MN - NM' = d^2x.$

$$\text{therefore } \lambda \cos \alpha = d^2x;$$

$$\text{whence } \cos \alpha = \rho \frac{d^2x}{ds^2}.$$

Similarly, $\cos \beta = \rho \dfrac{d^2y}{ds^2}$ $\cos \gamma = \rho \dfrac{d^2z}{ds^2}$.

Then from the relation $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$, we find

$$\frac{1}{\rho} = \sqrt{\frac{d^2x}{ds^2} + \frac{d^2y}{ds^2} + \frac{d^2z}{ds^2}}$$

which is the well known expression.

From these values of $\cos \alpha$, $\cos \beta$, $\cos \gamma$, Cauchy's demonstration of Meusnier's Theorem follows very readily.

Let $u = 0$ be the equation to a surface. Differentiating twice with regard to x, y, z, we have

$$0 = \frac{du}{dx} dx + \frac{du}{dy} dy + \frac{du}{dz} dz$$

$$0 = \frac{du}{dx} d^2x + \frac{du}{dy} d^2y + \frac{du}{dz} d^2z + \&c.$$

The terms we have left out depend only on the values of dx, dy, dz, and therefore only on the *direction* in which these variations are taken, and will be constant if we suppose them taken along a section of the surface made by a plane whose intersection with the tangent plane is constant. If ρ be the radius of curvature of the section, α, β, γ, its inclinations to the axes, we have

$$d^2x = \frac{ds^2}{\rho} . \cos \alpha, \&c.$$

Substituting these values

$$0 = \frac{1}{\rho} . \left(\cos \alpha \frac{du}{dx} + \cos \beta \frac{du}{dy} + \cos \gamma \frac{du}{dz} \right) + \text{const.}$$

the cosines of the angles which a normal to the surface makes with the axes are proportional to $\dfrac{du}{dx}, \dfrac{du}{dy}, \dfrac{du}{dz}$. Hence, calling θ the angle between the normal and the radius of curvature, $\dfrac{\cos \theta}{\rho} = \text{const.}$ When $\theta = 0$, ρ attains its maximum value $= R$.

Hence $$\rho = R \cos \theta,$$
or the radius of curvature of an oblique section is the projection of the radius of curvature of the normal section which has the same intersection with the tangent plane.

H. M.

MATHEMATICAL NOTES.

1. Since the publication of our first Number, we have met with the following proof, by Professor Jacobi of Koenigsberg, of the equation (K) page 16, which is better than the one there given.

As there, we have

$$\Gamma(m) \cdot \Gamma(n) = \int_0^\infty \int_0^\infty \varepsilon^{-x-y} x^{m-1} y^{n-1} dx\, dy.$$

Assume

$$x + y = r, \quad x = rz,$$

then, while x and y vary from 0 to ∞, r will vary from 0 to ∞, and since $z = \frac{x}{x+y}$, z will vary from 0 to 1. Suppose that x and z, y and r, vary together, then

$$dx = rdz, \quad dy = dr.$$

Substituting these values in the double integral, it becomes

$$\int_0^\infty \varepsilon^{-r} r^{m+n-1} dr \int_0^1 (1-z)^{n-1} z^{m-1} dz,$$

$$= \Gamma(m+n) \cdot \left(\frac{m}{n}\right),$$

whence $$\frac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m+n)} = \left(\frac{m}{n}\right).$$

γ.

2. *Diagonal of a parallelopiped.* The following is an easy method of obtaining the expression for the diagonal of a parallelopiped in terms of the edges, and the angles they make with each other.

Let a, b, c be the edges, r the diagonal.

α, β, γ the angles the edges make with each other.

λ, μ, ν the angles the diagonal makes with the edges.

Then projecting the three edges on the diagonal, we have

$$r = a \cos \lambda + b \cos \mu + c \cos \nu.$$

Again, if we project r on a, and compare it with the projection of the broken line a, b, c, which joins its extreme points, on the same line we have the relation

$$r \cos \lambda = a + b \cos \gamma + c \cos \beta.$$

Similarly,

$$r \cos \mu = a \cos \gamma + b + c \cos a,$$

and $$r \cos \nu = a \cos \beta + b \cos a + c.$$

Multiply by a, b, c, and add. Then, by the first condition,

$$r^2 = a^2 + b^2 + c^2 + 2ab \cos \gamma + 2ac \cos \beta + 2bc \cos a.$$

σ.

3. *In the Ellipse* $a'^2 + b'^2 = a^2 + b^2$. This well-known proposition may be very conveniently proved in the following manner. The equation to a diameter conjugate to that passing through (x', y') is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 0.$$

Transposing and squaring,

$$\frac{x^2x'^2}{a^4} = \frac{y^2y'^2}{b^4}.$$

Also, $y'^2 = \frac{b^2}{a^2}(a^2 - x'^2)$, and at the point where the diameter meets the curve $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$. Substituting these values, the equation becomes

$$\frac{x^2x'^2}{a^4} = 1 - \frac{(x^2 + x'^2)}{a^2} + \frac{x^2x'^2}{a^4},$$

$$\text{or } a^2 = x^2 + x'^2.$$

Similarly, $$b^2 = y^2 + y'^2;$$

therefore $$a^2 + b^2 = x^2 + y^2 + x'^2 + y'^2 = a'^2 + b'^2.$$

a.

4. *Problem in the papers for* 1838. A quadrant of an ellipse is inscribed in a right-angled triangle having its axes coincident with the sides containing the right angle. Find the curve traced out by its centre of curvature at the point of contact with the hypothenuse, while the point of contact moves from one acute angle to the other.

Let the origin be at the right angle, and let the sides be m, n, and the axes a, b; xy the coordinates of the point of contact, α, β of the centre of curvature. The equation to the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

Since the hypothenuse is a tangent

$$m = \frac{a^2}{x}, \quad n = \frac{b^2}{y},$$

$$\text{whence } \frac{a^2}{m^2} + \frac{b^2}{n^2} = 1.$$

Also, we have

$$\frac{x^3}{a^3} = \frac{a\alpha}{a^2 - b^2}, \quad \frac{y^3}{b^3} = -\frac{b\beta}{a^2 - b^2},$$

$$\text{therefore } \frac{a^3}{m^3} = \frac{a\alpha}{a^2 - b^2}, \quad \frac{b^3}{n^3} = -\frac{b\beta}{a^2 - b^2},$$

$$\text{whence } \frac{a^2}{m^2}(a^2 - b^2) = m\alpha, \quad \frac{b^2}{n^2}(a^2 - b^2) = -n\beta.$$

Adding these,

$$\left(\frac{a^2}{m^2} + \frac{b^2}{n^2}\right)(a^2 - b^2) = a^2 - b^2 = m\alpha - n\beta.$$

Again, multiplying the first by m^2 and the second by n^2, and subtracting,

$$(a^2 - b^2)^2 = m^3\alpha + n^3\beta,$$

$$\text{whence } m^3\alpha + n^3\beta = (m\alpha - n\beta)^2,$$

the required locus, which is a parabola.

π.

5. The area of the parallelogram formed by tangents applied at the extremities of any two conjugate diameters of an ellipse is constant.

Let x, y be the coordinates of the extremity of one diameter, and x', y' those of the other; θ, θ' the angles which they make with x. Then $x = a' \cos\theta$, $y = a' \sin\theta$, $x' = b' \cos\theta'$, $y' = b' \sin\theta'$.

The tangent whose inclination to the axis of x is θ, passes through the point $x'y'$: therefore, by the equation in page 9, substituting $\tan\theta$ for a, and multiplying by $\cos\theta$,

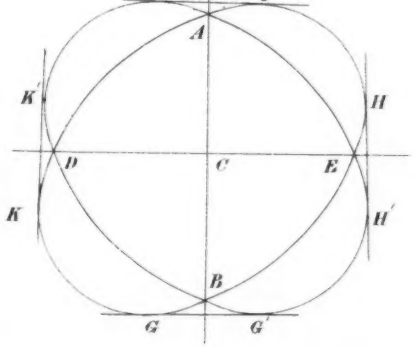
$$y' \cos\theta - x' \sin\theta = \sqrt{(b\cos\theta)^2 + (a\sin\theta)^2}.$$

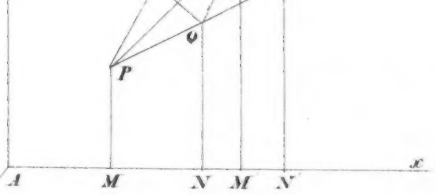
Multiply by a', and substitute for x' and y' their values in b' and θ', therefore

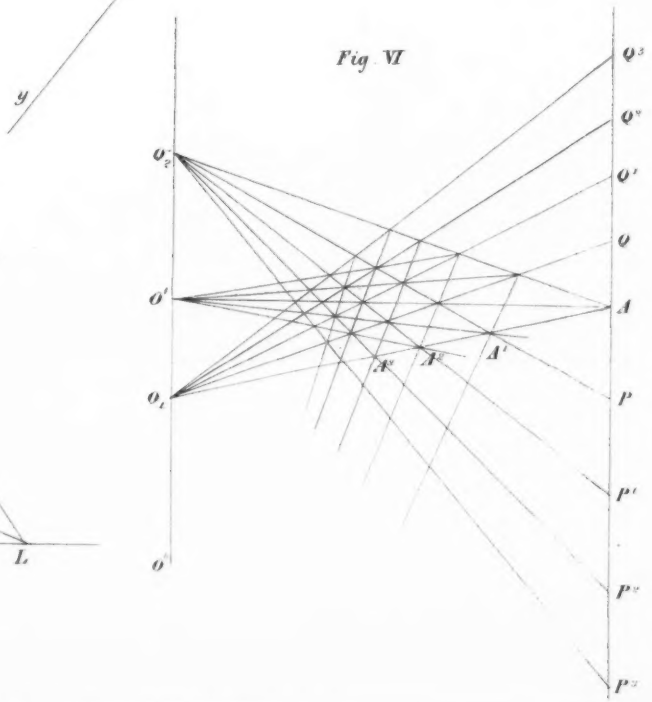
$$a'b' \sin(\theta' - \theta) = \sqrt{b^2(a'\cos\theta)^2 + a^2(a'\sin\theta)^2}$$

$$= \sqrt{b^2x^2 + a^2y^2} = ab.$$

γ.







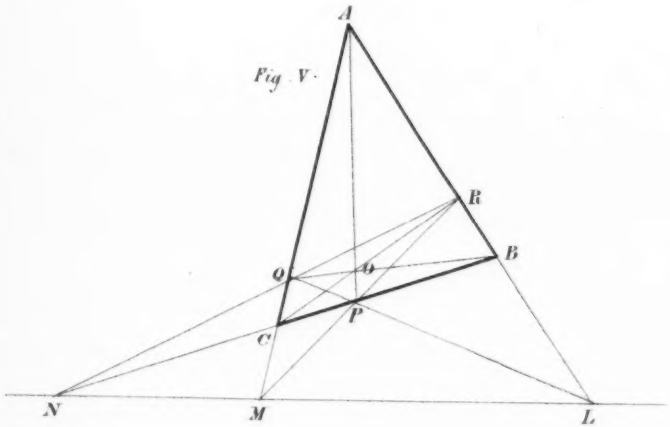


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Fig. I

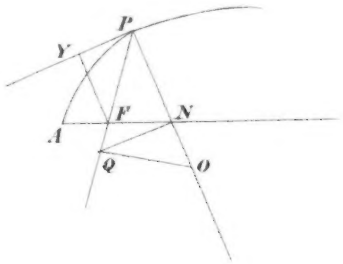

Fig. II

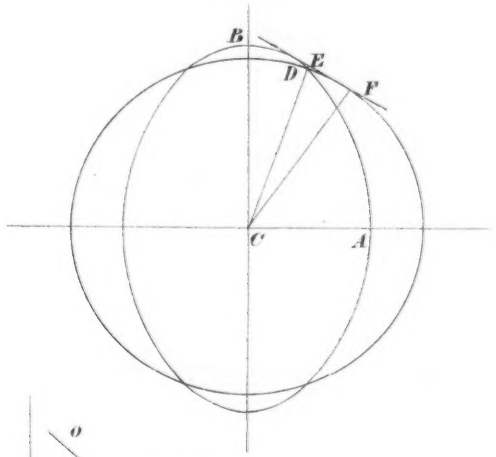

Fig. III

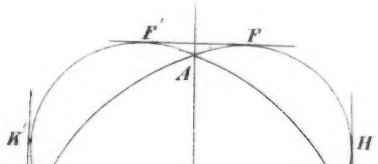
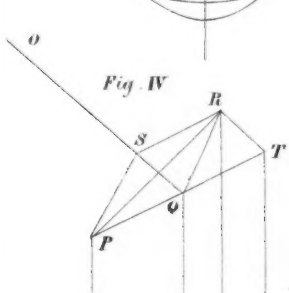
Fig. IV